Certifying Lemons
with
Discernible Hard Information

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(Preliminary, comments welcomed)

Abstract

A sender with private information (high or low ability) tries to convince a receiver of having higher ability. A certifier offers a menu of (blackwell) experiments and prices to screen the sender. The sender uses the experiment's outcome to persuade the receiver to take a favorable action. This paper focuses on the equilibrium interaction in this certification game when the receiver can distinguish between outcomes of the experiment only based on the hard information contained in the outcome. The main result characterizes all possible equilibrium outcomes in terms of a convex combination of menus containing only simple experiments. Using this characterization, I show the existence of an equilibrium in which soft information overrules hard information; due to equilibrium self-selection of the sender, some outcomes whose hard information makes the receiver more pessimistic about the sender's ability end up persuading the receiver to choose the favorable action.
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Introduction

Consider a simple adverse selection environment in which a privately informed sender (either high ability or low ability) tries to convince a receiver that he is high ability. The

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receiver chooses between one of two actions and has state-dependent preferences. When the receiver's prior belief about the sender's expected ability is low enough, she prefers to act unfavorably to the sender. Suppose a statistical procedure can evaluate the sender's private information. In that case, certification intermediaries can enable signaling between the sender and the receiver, allowing the sender to persuade the receiver to take the favorable action. Such intermediaries are present in various sectors, including research laboratories, consultants, auditors, and academic testing.[1]

This paper focuses on the role of discernible hard information in shaping the certification market. Hard information is *discernible* when the receiver can distinguish between experimental outcomes only based on their hard information. The exact procedure of the experiments might differ, but as long as their outcomes have the same statistical (hard) information, the receiver can not distinguish the experiments. Discernibility presents itself as a natural restriction on the certifier's communication abilities.[2]

Consider a revenue-maximizing certifier who offers a menu of experiments and price pairs. The design of this menu screens the sender for price discrimination. The experiments must be informative enough to persuade the receiver; the sender is only willing to pay for an experiment if it leads to a favorable action by the receiver. This restricts the informativeness of experiments offered by the certifier by restricting the certifier's ability to pool high and low ability senders. Discernibility of information further restricts the certifier from pooling in test outcomes with different hard information. Discernibility also prevents the certifier from segmenting the sender across experiment outcomes with the same hard information.

My main result, Theorem 1, characterizes all possible equilibrium outcomes in terms of a simple class of certificates that are offered in equilibrium. This shows the relationship between the equilibrium welfare and the design of optimal experiments. I use this characterization to establish two testable implications of my model. Equilibrium of the certification game might give rise to evidence generated by an experiment, which leads to a favorable action by the receiver even when the (statistical) hard information of the evidence leads to the receiver being more pessimistic (relative to her prior) about the sender's ability. Additionally, I show the existence of a separating equilibrium without

[1] Blair et al. (2011) discusses the potential impact of third-party certification on global commerce and discusses the need for regulation of the assurance and certification industry.

[2] Standards such as ANSI/ASQ Z1. 4 and ISO 2859 provide guidelines for the acceptance and sampling of products. These and many other standards essentially take the form of statistical tests.

perfectly informative experiments and the existence of partial pooling equilibria.

1 Model

I study a stylized model of a sender (he) and a receiver (she). The sender is privately informed about his binary ability, high or low, and this represents the sender's private type θ. The receiver decides to accept or reject the sender based on verifiable results of some (statistical/blackwell) experiment. The sender acquires these tests from a monopolist certifier. The certifier and receiver have a common prior μ about the sender's type. The certifier can flexibly post a menu of experiments and prices.[3] In particular, the certifier can send messages conditional on both the report of the sender and the sender's true type. The ability to condition on the sender's true type represents the certifier's "expertise"; the certifier has some ability or technology to evaluate the agent's private information. The certifier's signals are afforded credibility through some physical, contractual, or reputational reason.

1.1 Timing

t=1: The monopolist certifier posts a menu of experiments and prices observable to the sender and receiver.
t=2: The sender privately observes his type.
t=3: The sender privately purchases an option from the menu offered, or decides not to get certified.
t=4: The sender and receiver observe the realization of the purchased experiment. If no experiment was purchased, the receiver observes that the sender is not certified.
t=5: The receiver chooses an action in $A = \{a_h, a_l\}$.

The sender's type $\theta \in \{h, l\}$ represents his private knowledge about his ability. The receiver has a prior μ for the state being h. The receiver chooses between actions in $A = \{a_h, a_l\}$. The receiver's utility is $v : A \times \{h, l\} \to \mathbb{R}$ such that $v(a_h, h) > v(a_l, h)$ and $v(a_l, l) > v(a_h, l)$. The sender has a state independent utility $u : A \to \mathbb{R}$ such that $0 = u(a_l) < u(a_h) = 1$. The receiver's optimal choice, given his beliefs μ, is $a^*(\mu) = a_h$ if $\mu \geq \pi^*$ and a_l otherwise. Where $\pi^* = \frac{v(a_l,l)-v(a_h,l)}{v(a_l,l)-v(a_h,l)+v(a_h,h)-v(a_l,h)}$. I assume $0 < \mu < \pi^*$; the

[3] I don't allow the pricing to be contingent on the realization of the experiment.

receiver is ex-ante pessimistic about the sender's ability and will choose the unfavorable action a_l without any information. In particular, this assumption implies that the receiver only chooses a_h if the certifier generates some hard information.

The certifier acts like a mediator between the sender and receiver. The certifier publicly announces a menu of experiments and price pairs. The sender is privately charged a fee based on the experiment chosen from he menu. The certifier then publicly announces the realized message based on the true and reported type. I restrict attention to the certifier setting an upfront fee instead of a fee conditional on the realized signal[4]. The sender verifiably reports the realization of a statistical experiment.[5,6]

When information is discernible, each outcome can be represented by its likelihood ratio. Formally an experiment is given by $\sigma : \{h, l\} \to \Delta([0, \infty])$. We identify each signal realization $e \in [0, \infty]$ for the experiment σ with its corresponding likelihood ratio $\frac{d\sigma(e|h)}{d\sigma(e|l)} \in [0, \infty]$.[7] I refer to the realization $e \in [0, \infty]$ of the experiment σ as evidence or signal realized by σ. The signal realization e can be verifiably disclosed even if the experiment σ is unknown.[8]

For any report $\hat{\theta} \in \{h, l\}$ and signal $e \in [0, \infty]$, the probability of realization e conditional on the true type θ, i.e. $\sigma_{\hat{\theta}}(e|l)$ is a function of e and $\sigma_{\hat{\theta}}(e|h)$. Let Σ represent the set of all experiments.

1.2 Strategies

The certifier posts a menu of prices and experiments $m = \{(\sigma_i, \rho_i)_{i \in I}, (\Phi, 0)\}$. Where $\sigma_i : \{h, l\} \to \Delta([0, \infty])$ and $\rho_i \in [0, 1]$ and I is some arbitrary indexing set. Note for later that restricting the certifier to two non-trivial menu options is without loss, as the sender's private information is binary. The sender has the option not to buy any of the

[4]Allowing a fee conditional on the signal realization, enables the certifier to charge a fee that's correlated to the true state. The work of Faure-Grimaud et al. (2009) considers a certification model with outcome contingent fees, in particular, they show full surplus extraction by the certifier.

[5]The verifiability can be seen as a consequence of the certifier staking their reputation on the claim or some contractual restriction.

[6]For a topological space X, the set $\Delta(X)$ is the set of all Borel probability measures of X.

[7]Note that when restricted to $[0, \infty)$ absolute continuity holds, $\sigma(.|h) \ll \sigma(.|l)$. Thus $\frac{d\sigma(x|h)}{d\sigma(x|l)}$ is well defined in this restricted set. I extend the definition to set $[0, \infty]$ by requiring $\frac{\cdot}{0} := \infty$ and $\frac{0}{0} := 0$.

[8]The restriction on the message space to $[0, \infty]$ is not essential. We can allow for any message space that is homeomorphic to $[0, \infty]$. But using $[0, \infty]$ as the message provides for a convenient representation of experiments.

offered experiments. Thus, I require every menu to include a "no certification" signal (Φ) at 0 cost.[9] The set of all menus is $\mathcal{M}$.

The receiver's strategy (decison rule) is to choose an action $a \in \{a_h, a_l\}$ given some $m \in \mathcal{M}$ and $e \in ([0,\infty] \cup \Phi)$; this is given by a measurable function $\zeta : \mathcal{M} \times ([0,\infty] \cup \Phi) \to \{a_h, a_l\}$. Let $E_m^{\zeta} := \{e \in [0,\infty] \cup \Phi \mid \zeta(m,e) = a_h\}$. This represents the set of all evidence that lead to an action a_h when the certifier's menu is m and the receiver's decision rule is ζ. Each of the receiver's strategies (bijectively) corresponds to a collection $(E_m^{\zeta})_{m \in \mathcal{M}}$. Whenever convenient, I will omit writing ζ as part of the receiver strategy and express it in terms of a collection of sets $(E_m)_{m \in \mathcal{M}}$.

The sender's strategy is a selection rule that specifies the menu item chosen by the sender given his type θ, and the posted menu $m \in \mathcal{M}$. The selection strategy is given by $\widetilde{\gamma}_\theta(m) \in m$ and $\widetilde{\gamma}_\theta(m) = (\ \gamma_\theta(m), p(\gamma_\theta(m))\ )$. The sender's payoff from a selection γ_θ when he faces a menu m and anticipates the acceptance set E is $\int_E d\gamma_\theta(m)(e|\theta) - p(\gamma_\theta(m))$.

1.3 Histories

The certifier's history is the empty history $h_{ms} = \varnothing$. The sender moves at a history $h_s = (m, \theta) \in \mathcal{M} \times \{h, l\}$. The receiver's history is $h_r = (m, e) \in \mathcal{M} \times ([0,\infty] \cup \Phi)$.

2 Equilibrium

The solution concept is PBE in pure strategy with tie-breaking assumptions.[10] An **equilibrium strategy profile** of the game is given by the tuple $((E_m)_{m \in \mathcal{M}}, \gamma^*, m^*)$ which satisfies the following:

- **Certifier rationality:** $m^* \in \arg\max_{m \in \mathcal{M}} \mathbb{E}_\mu[p(\gamma_\theta^*(m))]$
- **Sender rationality:** for all $m = \{(\sigma_i, \rho_i)_{i \in I}, (\Phi, 0)\}$ the sender's selection rule:

$$\gamma_\theta^*(m) \in \arg\max_{(\sigma', p(\sigma')) \in m} \int_{E_m} d\sigma'(e|\theta) - p'$$

- **Bayes rule where possible:**

[9] More precisely, each menu contains a deterministic experiment that takes value Φ.

[10] I focus only on pure strategies. Moreover, I require in equilibrium the receiver always breaks ties in favor of the sender

– At the history (m, e) such that $e \in \text{supp}(\gamma^*_\theta(m)(.|\theta))$ for some type $\theta \in \{h, l\}$, the receiver's belief about the sender's type being high is given by Bayes rule:

$$\mu_{m,e} = \frac{d\gamma^*_h(m)(e|h)\mu}{d\gamma^*_h(m)(e|h)\mu + d\gamma^*_l(m)(e|l)(1-\mu)}$$

– At history (m, e) such that $e \notin \text{supp}(\gamma^*_\theta(m)(.|\theta))$ for both types $\theta \in \{h, l\}$. The receiver has arbitrary beliefs $\mu_{m,e}$ about the sender's type.

- **Receiver rationality:** for any $m \in \mathcal{M}$, the receiver's acceptance set satisfies:

$$E_m = \{e \in [0, \infty] \cup \Phi | a_h \in \underset{a \in \{a_l, a_h\}}{\arg\max} \mathbb{E}_{\mu_{m,e}}[v(a, \theta)]\}$$

Let $\overline{\mathcal{E}}$ represent the set of all such equilibria. I refer to E_{m^*} as the receiver's equilibrium (or on-path) **acceptance set**.[11]

Note that given $(E_m)_{m \in \mathcal{M}}$, sequential rationality and tie-breaking uniquely pin down γ^*, thus whenever convenient I will omit explicitly mentioning the sender's equilibrium selection strategy γ^*.

Define $(E_{m^*}, m^*) := ((E'_m)_{m \in \mathcal{M}}, \gamma^*, m^*)$ where $E'_m = \varnothing$ when $m \neq m^*$ and $E'_{m^*} = E_{m^*}$.

Lemma 1. *If $((E_m)_{m \in \mathcal{M}}, \gamma^*, m^*) \in \overline{\mathcal{E}}$ then $(E_{m^*}, m^*) \in \overline{\mathcal{E}}$.*

Proof. The last three conditions in the definition of equilibrium are satisfied as $((E_m)_{m \in \mathcal{M}}, \gamma^*, m^*) \in \overline{\mathcal{E}}$. By definition of the off-path actions of the receiver E'_m, it's immediate that certifier rationality for m^* also holds. □

Essentially, the receiver can hold arbitrarily pessimistic beliefs following a deviation by the certifier. This trivially makes m^* the best response of the certifier. By Lemma 1, we know m is offered on-path in some equilibrium (in $\overline{\mathcal{E}}$) along with the receiver's on-path acceptance set E_m if and only if

$$\frac{\int_{E'} d\gamma^*_h(m)(e|h)}{\int_{E'} d\gamma^*_l(m)(e|l)} \geq \frac{1}{l(\mu)} \qquad \forall\, E' \subset E_m$$

Where $l(\mu) = \frac{\mu(1-\pi^*)}{\pi^*(1-\mu)}$, i.e. $l(\mu)$ is the minimum likelihood ration, given prior μ, that is needed for a_h to be sequentially rational response of the receiver.

A menu $m^* \in \mathcal{M}$ is **valid** with respect to E if $(E, m) \in \overline{\mathcal{E}}$. Where $(E, m) = ((E_{m'})_{m' \in \mathcal{M}}, \gamma^*, m)$ such that $E_m = \varnothing$ whenever $m \neq m^*$ and $E_{m^*} = E$.

[11] A stronger version of PBE might require the element $e = \infty \in E_m$ for every $m \in \mathcal{M}$ Such a restriction could represent aspects of objectiveness in the evidence produced by the certifier. This affects none of the results from section 4 onwards, so I won't impose this restriction.

Remark 1. An immediate conclusion from receivers ex-ante pessimism ($\mu < \pi^*$) is that for any equilibrium $((E_m)_{m\in\mathcal{M}}, \gamma^*, m^*) \in \overline{\mathcal{E}}$ it must be that $\Phi \notin E_m$ for all $m \in \mathcal{M}$. This means that not getting certified never leads to acceptance by the receiver in any equilibrium of the game. This is easy to see, as if $\Phi \in E_m$, then at the subgame following the menu choice m of the certifier, either the sender or the receiver has a profitable deviation. Thus, in the rest of the paper, all acceptance sets E are assumed to be such that $\Phi \notin E$.

The sender's private information is binary, hence it is without loss to consider menus with at most two (non-trivial) informative experiments.

Fix some set $E \subset [0,\infty]$, the menu $m = \{(\sigma_\theta^m, \rho_\theta^m)_{\theta\in\{h,l\}}, (\Phi, 0)\} \in \mathcal{M}$ is **obedient** with respect to E if m satisfies the following:

- **Sender IC and IR**

$$\int_E d\sigma_\theta^m(e|\theta) - \rho_\theta \geq \int_E d\sigma_{\theta'}^m(e|\theta) - \rho_{\theta'} \qquad \forall\, \theta, \theta' \in \{l, h\}$$

$$\int_E d\sigma_\theta^m(e|\theta) \geq \rho_\theta^m \qquad \forall\, \theta \in \{l, h\}$$

- **Receiver obedience**

$$\frac{\int_{E'} d\sigma_h^m(e|h)}{\int_{E'} d\sigma_l^m(e|l)} \geq \frac{1}{l(\mu)} \qquad \forall\, E' \subset E$$

The certifier's revenue from an obedient menu m be given by $\text{rev}(m) = \mu\rho_h^m + (1-\mu)\rho_l^m$. For each $E \subset [0,\infty] \cap \Phi$, the set of menus obedient with respect to E is given by $\mathcal{M}_E$.

Two equilibria $((E_m)_{m\in\mathcal{M}}, m^*)$ and $((E'_m)_{m\in\mathcal{M}}, m')$ are **outcome equivalent** if:

- $\gamma_\theta^*(m^*)(.|h)|_{E_{m^*}} = \gamma_\theta^*(m')(.|h)|_{E'_{m'}} \;\; \forall\;\theta$ (Experiment selection)
- $\mathbb{E}_\mu[p(\gamma_\theta^*(m^*)] = \mathbb{E}_\mu[p(\gamma_\theta^*(m')]$ (Revenue)

The first condition requires that in both equilibria the conditional distribution of signals restricted to the receiver's equilibrium acceptance set is the same.[12]

The second condition requires that the certifier earns the same revenue in both equilibria. In particular, these conditions imply that $E_{m^*} = E'_{m'}$. Moreover, whenever the equilibrium menus m^*, m' are obedient with respect to E_{m^*} and $E'_{m'}$ respectively then

[12]When the distribution of signals that lead to acceptance by the receiver is estimatable by some external researchers, the certification menu allows the researcher to infer facts about the equilibrium based on (statistical) hard information of an experiment's outcomes.

$\sigma_\theta^{m^*}|_{E_{m^*}} = \sigma_\theta^{m'}|_{E'_{m'}}$. As defined, outcome equivalence implies payoff equivalence but is not implied by payoff equivalence. If two equilibrium outcomes are equivalent, then the distribution of messages conditional on the receiver accepting is the same across the equilibria for each type of sender.

The following observation highlights the key role played by obedient menus.

Observation 1. If a menu $m^* \in \mathcal{M}$ is offered in equilibrium, then there exists an outcome equivalent equilibrium $((E')_{m\in\mathcal{M}}, \gamma', m')$ such that $m' \in \mathcal{M}_{E'_{m'}}$. If a menu m^* is obedient with respect to some $E \subset [0, \infty]$, then an equilibrium exists in which the on-path menu is m^*.

Proof. See appendix. □

Remark 2. Observation 1 is essentially the revelation principle with a restriction on the messages that the certifier can send to the receiver. This restriction results from the assumption that the receiver can distinguish between experiments' outcomes if and only if the outcomes have different likelihood ratios (hard information). In particular, the certifier can not pool evidence e with evidence $e' \neq e$ into a single message sent to the receiver.

Lemma 2. *Any equilibrium $((E_m)_{m\in\mathcal{M}}, \gamma^*, m^*) \in \overline{\mathcal{E}}$ in which $e = 0 \in E^*_m$ is outcome equivalent to $((E'_m)_{m\in\mathcal{M}}, \gamma' m') \in \overline{\mathcal{E}}$, where $E'_m = E_m$ for $m \neq m^*$, $E'_{m^*} = E_{m^*} \setminus \{0\}$, $m' = m^*$ and $\gamma' = \gamma^*$.*

Proof. Whenever $e = 0 \in E_{m^*}$, the signal $e = 0$ has probability 0 of being disclosed in equilibrium. Thus, removing $e = 0$ from E_{m^*} does not affect the equilibrium outcomes. □

Let the set of equilibria in which the on-path menu is obedient and $e = 0$ is not part of the receiver's equilibrium strategy E_{m^*} be given by $\mathcal{E}$. As I am interested in the equilibrium outcomes, it is without loss to restrict attention to equilibrium in $\mathcal{E}$. In usual mechanism design terms (Forges (1986), Myerson (1986)), observation 1 shows that restricting to truthful mechanisms is enough, but as outcome equivalence is stronger than payoff equivalence, restricting to a direct recommendation mechanism (pass-fail experiments) is not necessarily without loss for the hard information environment in mind.

3 Refinement

Observation 1 shows that any menu that is obedient with respect to some E is offered on-path in some equilibrium of the game. This conclusion relies on allowing the receiver to have arbitrary off-path beliefs. To focus on equilibria where the on-path menu is revenue maximizing with respect to the receiver's on-path acceptance set E, I propose a refinement of the receiver's off-path beliefs which selects for such equilibria.

The proposed refinement imposes a form of consistency on how the receiver evaluates evidence following the certifier's deviation. Essentially, the receiver does not change his acceptance set unless there is a compelling reason to do so.

Refinement (Consistent Evaluation): Let the set of refined equilibrium be given by:

$$\overline{\mathcal{E}_r} := \{((E_m)_{m\in\mathcal{M}}, \gamma^*, m^*) \in \overline{\mathcal{E}} \mid \forall\, m \in \mathcal{M},\ (E_{m^*}, m) \in \overline{\mathcal{E}} \implies E_m = E_{m^*}\}$$

Recall for any aceptence set E and menu m the collection $(E, m) = ((E'_{m'})_{m'\in\mathcal{M}}, \gamma^*, m)$ where $E'_{m'} = \varnothing$ when $m' \neq m$ and $E'_m = E$.

Consider an equilibrium $((E_m)_{m\in\mathcal{M}}, \gamma^*, m^*)$. If m^*, E_{m^*} are the on-path menu and receiver's acceptance set, then after observing a menu m valid with respect to E_{m^*}, the receiver is sequentially rational to use acceptance set E_{m^*} given that the sender' choice is in $\gamma^*(m, E_{m^*})$. Moreover, the type θ sender anticipates this and chooses $\sigma \in \gamma^*_\theta(m, E_{m^*})$. However, if a menu m is not valid with respect to E_{m^*} then using the acceptance set E_{m^*} is not rational for the receiver, given the sender's choice is $\gamma^*(m, E_{m^*})$. In this case, the refinement does not restrict the sender's and receiver's off-path beliefs. Thus, the refinement requires that, following the certifier's deviation, the receiver does not change his acceptance set whenever it is sequentially rational to do so, given that the sender anticipates facing the on-path acceptance set.

The proposed refinement selects for equilibria in which both the sender and receiver form consistent expectations about outcomes, even after an observable deviation in the testing structure. A particular consequence is that the refinement rules out outcomes in which the certifier gets less than second-best payoff, fixing the receiver's on-path strategy.

Let $\mathcal{M}^r_E$ be the set of all menus that maximize the certifier's revenue among all $m \in \mathcal{M}$ that are obedient with respect to $E \subset (0, \infty]$. I refer to some $m \in \mathcal{M}^r_E$ as a revenue-maximizing menu with respect to E. By applying observation 1 (and Lemma 2), we can define $\mathcal{E}_r \subset \overline{\mathcal{E}_r}$ to be the set of equilibrium such that the equilibrium menu $m^* \in \mathcal{M}_{E_{m^*}}$, where E_{m^*} is the receiver's equilibrium acceptance set. More precisely, the set of equilibria $\mathcal{E}$ is such that for any equilibrium in $\mathcal{E}$, the on-path menu is obedient with respect to the receiver's on-path acceptance set and $e = 0$ is not part of the receiver's on-path acceptance set. The following lemma demonstrates the property of $\mathcal{E}_r$ eluded in the last paragraph.

Lemma 3. *If $((E_m)_{m\in\mathcal{M}}), m^*) \in \mathcal{E}_r$ then $m^* \in \mathcal{M}^r_{E_{m^*}}$. Moreover, if $m^* \in \mathcal{M}^r_E$ for some $E \subset [0, \infty]$ then there exists an equilibrium in $\mathcal{E}_r$ such that m^* is offered on-path and receiver's on-path acceptance set is E.*

Proof. See appendix □

Remark 3. If two equilibria $((E_m)_{\mathcal{M}}, \gamma, m^*)$, $((E'_m)_{\mathcal{M}}, \gamma', m') \in \mathcal{E}_r$ with $E_{m^*} = E'_{m'}$ are such that $\sigma^{m^*}_\theta$ and $\sigma^{m'}_\theta$ only differ on $E^C_{m^*}$, then the two equilibria are outcome equivalent.

4 Equilibrium Outcomes

In this section, I will focus on equilibrium in $\mathcal{E}_r$ such that on-path $e = 1 \notin E_{m^*}$. These equilibria constitute informative equilibria.

4.1 Structure of Equilibrium Certificates

This section establishes that the equilibria in $\mathcal{E}_r$ can be studied by solving a constrained linear optimization problem. I characterize all possible equilibrium outcomes in $\mathcal{E}_r$ by first characterizing $\mathcal{M}^r_E$ in terms of a simplified linear optimization problem. Using this, I show that a simpler set of menus generates all solutions to the optimization problem. Finally, I find the optimal menus among these simple menus.

Proposition 1. *Fix some $E \subset (0, \infty]$. Let $m^* = ((\sigma^{m^*}_\theta, \rho^{m^*}_\theta)_{\theta\in\{l,h\}})$ be obedient wrt E. A menu $m^* \in \mathcal{M}^r_E$ if and only if m^* is such that*

$$\rho^{m^*}_h = \int_E \left[d\sigma^{m^*}_h(e|h) - \left(1 - \frac{1}{e}\right) d\sigma^{m^*}_l(e|h) \right]$$

,

$$\rho^{m^*}_l = \int_E d\sigma^{m^*}_l(e|l)$$

and solves the following optimization problem:

$$\max_{(\sigma^m_h(.|h), \sigma^m_l(.|h)) \in \Delta([0,\infty]) \times \Delta([0,\infty])} \mu \int_E d\sigma^m_h(e|h) + \int_E \left(\frac{1}{e} - \mu\right) d\sigma^m_l(e|h)$$

subject to

$$\int_E \left(1-\frac{1}{e}\right) d\sigma_h^m(e|h) \geq \int_E \left(1-\frac{1}{e}\right) d\sigma_l^m(e|h) \geq 0$$

$$\frac{\int_{E'} d\sigma_h^m(e|h)}{\int_{E'} d\sigma_l^m(e|l)} \geq \frac{1}{l(\mu)} \text{ for all } E' \subset E \cup \operatorname{supp}(\sigma_l(.|l))$$

Proof. See appendix □

Remark 4. The proposition follows by noting two properties of the optimal menu. First, it leaves zero rent to the low type. Second, the high type's willingness to pay for either of the offered experiments is weakly greater than the willingness to pay of the low type sender.

Remark 5. Note that $\int_E \left(1-\frac{1}{e}\right) d\sigma_h^m(e|h) \geq \int_E \left(1-\frac{1}{e}\right) d\sigma_l^m(e|h) \geq 0$ implies that $\sigma_\theta^m(E|l) < \sigma_\theta^m(E|h) \leq 1$. As we are interested in outcome equivalence, we can set $\sigma_\theta(0|l) = 1-\sigma_\theta(E|l)$, making the choice of experiments well defined.

Corollary 1. *If $m \in \mathcal{M}_E^r$ and $rev(m) > 0$ for some $E \subset [0,\infty]$ then $\int_E d\sigma_h^m(e|h) = 1$.*

Proof. See appendix □

This shows that any equilibrium menu in $\mathcal{E}_r$ leads to the high type being accepted with probability 1. In particular, the receiver faces no distortion whenever the sender is high type. But the receiver's payoff might be distorted when the sender is low type. Whenever $m \in \mathcal{M}_E$ for some E, information rent is only conceded to the high type. This rent is given by, $\text{rent}(m) = \int_E \left(1-\frac{1}{e}\right) d\sigma_l^m(e|h)$.

Another immediate consequence of Proposition 1 is the existence of a separating equilibrium. I call an equilibrium $((E_m)_{m\in\mathcal{M}}, m^*) \in \mathcal{E}$ **separating** whenever

$$\int_{E_{m^*}} d\sigma_h^{m^*}(e|h) = 1 \text{ and } \int_{E_{m^*}} d\sigma_l^{m^*}(e|h) = 0$$

Corollary 2. ***(Existence of separating equilibrium)*** *If an equilibrium $((E_m)_{m\in\mathcal{M}}), m^*) \in \mathcal{E}_r$ is separating then $e^* := \inf(E_{m^*}) \geq \frac{1}{\mu}$. Moreover, if for an equilibrium $((E_m)_{m\in\mathcal{M}}), m^*) \in \mathcal{E}_r$ it holds that $e^* := \inf(E_{m^*}) > \frac{1}{\mu}$, then the equilibrium is separating.*

The theorem shows the existence of separating equilibria when the on-path experiment is an imperfect quality certification (De and Nabar (1991) and Strausz (2010)). Although the high type sender is accepted with probability 1, the low type sender might also be accepted with positive probability, conditional on buying the experiment.

4.1.1 Equilibrium Characterization

In this section, I restrict attention to equilibrium with countable acceptance sets.[13]

For any $E \subset [0, \infty]$, let

$$\mathcal{T}(E) := \{m \in \mathcal{M}^r_E \mid |\operatorname{supp}(\sigma^m_h(.|h)) \cap E| \leq 3, \ |\operatorname{supp}(\sigma^m_l(.|h)) \cap E| \leq 2\}$$

Theorem 1. *If $((E_m)_{m\in\mathcal{M}}, m^*) \in \mathcal{E}_r$ such that E_{m^*} is countable, then there exists an outcome equivalent equilibrium $((E'_m)_{m\in\mathcal{M}}, m') \in \mathcal{E}_r$ such that $m' \in \boldsymbol{cvx}(\mathcal{T}(E_{m^*}))$.*[14]

Proof. See appendix □

The proof of the theorem involves solving the linear optimization problem in proposition 1, to do so, I first deal with the point-wise inequalities $\frac{\sigma_h(e|h)}{\sigma_l(e|l)} \geq \frac{1}{l(\mu)}$ for all $e \in E \cap \operatorname{supp}(\sigma_l(e|l))$. Then I proceed by solving the simplified problem by finding the extreme points of the feasible set of experiments. The theorem establishes that all equilibrium outcomes in $\mathcal{E}_r$ are generated by a simple class of menus. In particular, the theorem characterizes the implementable rent distributions for equilibrium in $\mathcal{E}_r$.

Corollary 3. ***(Soft information overrules hard information)*** *There exists equilibria in $\mathcal{E}_r$ such that after observing some evidence $e < 1$, the receiver chooses action a_h.*

Proof. See proof of Lemma 5 in the appendix. □

The corollary shows the existence of an equilibrium in which the receiver takes a favorable action even after observing evidence whose hard information makes the receiver more pessimistic about the sender.

Corollary 4. ***(Existence of partial pooling)*** *There exists equilibria in $\mathcal{E}_r$ such that* $\operatorname{supp}(\sigma_l) \cap E \subsetneq \operatorname{supp}(\sigma_h) \cap E$.

Proof. See appendix. □

Partial pooling equilibrium corresponds to the optimal menu offered by the certifier that consists of experiments with outcomes that only a high type can achieve with positive probability, and also outcomes that both high and low type senders can achieve with positive probability.

[13] I prove the statement in the appendix for discreet support distribution.

[14] For any set X the convex hull **cvx**(X) is defined as the smallest convex set containing X

5 Literature Review and Discussion

The issue of signaling and market efficiency in pure adverse selection has been widely studied in economic literature. The seminal paper byAkerlof (1970) shows how markets can unravel in the presence of adverse selection, Viscusi (1978) demonstrates that quality certification can provide an alternative to exiting the market for high type producers. Spence (1978) shows that undertaking costly actions can help signal private information in the context of labor markets.

This paper, like Lizzeri (1999), considers a monopolistic information intermediary, which can produce hard (verifiable) information about the sender's private type. Unlike models of signaling where the costly action undertaken for signaling is wasteful, here the costly action directly corresponds to the payoff of the intermediary. Like my model, Lizzeri (1999) also demonstrates that the benefit from the presence of a certifier when the receiver is pessimistic.

I consider both the information design and monopolist screening problem that the intermediary faces; Lizzeri (1999) only focuses on the design aspect in the case of an optimistic receiver. In Lizzeri (1999) optimal certification mechanism leads to a single uninformative experiment. Thus, the certifier's ability to generate hard information is moot, and so is the discernibility of hard information

The disclosure of hard evidence is often studied in the context of voluntary disclosure Grossman (1981), Grossman and Hart (1983), Milgrom (1981). In the context of this paper, the main takeaway from models of voluntary disclosure is the minimum principle (Guttman et al. (2014), DeMarzo et al. (2019)): non-disclosure is treated in the most pessimistic way by the receiver. The minimum principle holds in my model, as the prior belief ($\mu < \pi^*$) prescribes choosing a_l whenever the receiver observes no certification.

Although I focus on a single monopolist certifier, the analysis in the paper describes all possible equilibrium welfare. The different equilibrium outcomes can be interpreted as cases when the players have varying levels of market power which reduces the share of surplus that the monopolist can capture.

Previous works on monopolistic certification have studied the welfare implications in isolation. My work emphasizes the relationship between test design and welfare aspects in these environments. Notably, Weksler and Zik (2023) considers the test design and welfare in markets with monopolistic certifiers. However, unlike this paper Weksler and Zik (2023) does not study a monopolistic screening problem by only allowing the certifier to post a single experiment. The paper by Dasgupta et al. (2022) considers a general test design and screening problem similar to mine, but they focus on an environment where an uninformed sender can flexibly design a test before interacting with a receiver (first-party certification). The issue of test design is also studied by Ali et al. (2022), their analysis differs from mine as they are primarily concerned with outcomes when the intermediary worries about the worst-case revenue across all equilibrium outcomes.

Closely related to this paper is Corrao (2023), which considers a similar monopolistic certification problem but focuses on soft information. In Corrao (2023), the intermediary cannot condition the experiment's outcomes on the true type of the sender and must rely solely on the reported type. They show that the mediator when restricted to soft information, can induce any receiver's belief that is consistent with hard information. In my model, the only equilibrium outcome implementable by soft information is the certifier optimal outcome (section 5.1).

Following the work of Kleiner et al. (2021), it's common in economic theory to study outcomes of mechanism design and information design problems geometrically. The technical result of this paper follows this theme by first reducing the screening and design to a constrained linear optimization problem, then characterizing the equilibrium outcomes in terms of the extreme points of the feasible set of experiments offered by the certifier.

The literature on monopolistic certification also considers a moral hazard environment; Shapiro (1986), Albano and Lizzeri (2001), Zapechelnyuk (2020). However, similar to the certification literature with pure adverse selection, these papers are silent about the design of optimal tests. This presents a future direction for extending the methodology of this paper to other certification environments where the investment in costly action not only corresponds to the surplus of the certifier but also has a role in capital development for the receiver.

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6 Appendix

6.1 Useful lemmas

Lemma 4. *There exists $e > 1 \in E$ if and only if there exists a menu $m \in \mathcal{M}_E$ such that $rev(m) > 0$ (or equivalently $\int_E d\sigma_h^m(e|h) > 0$).*

Proof. Sufficiency: when $e > 1 \in E$, the menu m such that $\sigma_h^m(e|h) = 1$, $\rho_h^m = 1$ and $\sigma_l^m = \Phi$, $\rho_l^m = 0$ is obedient wrt E.
Necessity: first note that incentive compatibility implies the monotonicity of allocation:

$$\int_E [d\sigma_h^m(e|h) - d\sigma_l^m(e|h)] \geq \int_E \frac{1}{e}[d\sigma_h^m(e|h) - d\sigma_l^m(e|h)]$$

If $e \in E \implies e < 1$, we see that only the menu m with $\int_E d\sigma_h^m(e|h) = \int_E d\sigma_l^m(e|h)$ satisfy this. In particular, $\rho_h^m = \rho_l^m$. By obedience condition of the receiver we get that $\int_E d\sigma_h^m(e|h) \geq \int_E \frac{1}{el(\mu)} d\sigma_l^m(e|h)$. But this implies $\int_E d\sigma_l^m(e|h) \geq \int_E \frac{1}{el(\mu)} d\sigma_l^m(e|h)$. As $l(\mu) < 1$ and by assumption $e < 1$ for all $e \in E$, this implies $\int_E d\sigma_h^m(e|h) = \int_E d\sigma_l^m(e|h) = 0$. By individual rationality of the sender, we get $\rho_h^m = \rho_l^m = 0$ □

Corollary 5. *Let $m \in \mathcal{M}_E$ for some $E \subset [0, \infty]$, if there exists $E' \subset E \cap [0, 1]$ such that $\int_{E'} d\sigma_h^m(e|h) > 0$ then there exists some $E'' \subset E \cap [1, \infty]$ such that $e > 1$ and $\int_{E''} d\sigma_h^m(e|h) > 0$.*

Proof. Let there exists $E' \subset E \cap [0, 1]$ such that $\int_{E'} d\sigma_h^m(e|h) > 0$, if there doesn't exist $E'' \subset [0, \infty]$ with the properties mentioned above then $\int_E d\sigma_h^m(e|h) = 0$. This follows from reasoning similar to the necessity part of lemma 1; to avoid violation of monotonicity of allocation $\int_E d\sigma_h^m(e|h) = \int_E d\sigma_l^m(e|h)$. Then for the menu m to be obedient wrt E it must hold that $\int_E d\sigma_h^m(e|h) = \int_E d\sigma_l^m(e|h) = 0$. Thus the corollary holds by contradiction. □

6.2 Proof of Observation 1

Observation 1:

If a menu $m^* \in \mathcal{M}$ is offered in equilibrium, then there exists an outcome equivalent equilibrium $((E')_{m \in \mathcal{M}}, \gamma', m')$ such that $m' \in \mathcal{M}_{E'_{m'}}$. If a menu m^* is obedient wrt some $E \subset [0, \infty]$, then an equilibrium exists in which the certifier offers m^* on-path.

Proof. If a menu m^* is obedient wrt E, then $\gamma_\theta^*(m^*, E) = \sigma_\theta^{m^*}$ and $\mu_{m^*,e} \geq \pi^*$ for all $e \in E$. [15]These two then imply that (E, m^*) is in $\overline{\mathcal{E}}$. Pessimistic off-beliefs of the receiver imply

[15] (E, m^*) is defined in lemma 1 in the main text.

that the certifier has no incentive to deviate. The sender is best responding as the menu m^* and the receiver's strategy E, as m^* is obedient wrt E. Obedience implies that the receiver is best responding to Bayesian beliefs on-path given the equilibrium menu and the sender's selection rule. Thus $(E, m^*) \in \overline{\mathcal{E}}$.

Let $((E_m)_{m\in\mathcal{M}}, \gamma^*, m^*) \in \overline{\mathcal{E}}$. If m^* satisfies sender IC wrt E_{m^*} this is $\gamma^*_\theta(m, E) = \sigma_\theta^{m^*}$, then by the definition of equilibrium we get that m^* satisfies receiver obedience wrt E_{m^*}. Assume that m^* doesn't satisfy sender IC wrt E_{m^*}. This implies that for some $\theta \in \{h, l\}$ the senders selection $\gamma_\theta(m^*, E^*) \neq \sigma_\theta^{m^*}$. In this case the menu $m' := ((\gamma_\theta(m^*), p(\gamma_\theta(m^*)))_{\theta\in\{h,l\}}$ is obedient wrt E_{m^*}. Thus $(E_{m^*}, m') \in \overline{\mathcal{E}}$ by the first part of the proof. Moreover, the equilibrium (E_{m^*}, m') is outcome equivalent to $((E_m)_{m\in\mathcal{M}}, \gamma, m^*)$ by construction. □

6.3 Proof of Lemma 3

Lemma 3:

If the $((E_m)_{m\in\mathcal{M}}), m^*) \in \mathcal{E}_r$ then $m^* \in \mathcal{M}_E^r$. Moreover, if $m^* \in \mathcal{M}_E^r$. for some $E \subset [0, \infty]$ then there exists an equilibrium in $\mathcal{E}_r$ for which m^* is offered on-path and the on-path acceptance set is $E_{m^*} = E$.

Proof. If $((E_m)_{m\in\mathcal{M}}, m^*) \in \mathcal{E}_r$ then by definition $((E_m)_{m\in\mathcal{M}}, m^*) \in \mathcal{E}$ and $E_m = E_{m^*}$ for all $m \in \mathcal{M}_{E_{m^*}}$. Recall that if m^* is offered on-path then m^* maximizes the certifier's revenue among all menus $m \in \mathcal{M}$ given the sender's best responds to the receiver's strategy $(E_m)_{m\in\mathcal{M}}$. As $E_m = E_{m^*}$ for all $m \in \mathcal{M}_{E_{m^*}}$, we get that $\text{rev}(m^*) \geq \text{rev}(m)$ for all $m \in \mathcal{M}_{E_{m^*}}$. Thus $m^* \in \mathcal{M}^r_{E_{m^*}}$.

For the other direction note that if $m^* \in \mathcal{M}_E^r$ for some E then $(E, m^*) \in \mathcal{E}$. We construct the following equilibrium profile $((E_m)_{m\in\mathcal{M}}, m^*)$ where $E_m = \varnothing$ whenever $m \notin \mathcal{M}_E$ and $E_m = E$ whenever $m \in \mathcal{M}_E$. Now note that $((E_m)_{m\in\mathcal{M}}, m^*) \in \mathcal{E}$ as m^* is obedient wrt $E_{m^*} = E$, and by construction m^* maximizes the certifier's revenue given the sender's best response to the receiver's strategy. Thus $((E_m)_{m\in\mathcal{M}}, m^*) \in \mathcal{E}_r$. □

For the rest of the proofs, I restrict attention to acceptance sets E such that $e = 1 \notin E$.

6.4 Proof of Proposition 1

Proposition 1:

Fix some $E \subset (0, \infty]$. Let $m^* = ((\sigma_\theta^{m^*}, \rho_\theta^{m^*})_{\theta\in\{l,h\}})$ be obedient wrt E. Then $m^* \in \mathcal{M}_E^r$ if

and only if m^* is such that
$\rho_h^{m^*} = \int_E \left[d\sigma_h^{m^*}(e|h) - \left(1 - \frac{1}{e}\right) d\sigma_l^{m^*}(e|h)\right]$, $\rho_l^{m^*} = \int_E d\sigma_l^{m^*}(e|l)$ and solves the following optimization problem:

$$\max_{(\sigma_h^m(.|h),\sigma_l^m(.|h))\in\Delta([0,\infty])\times\Delta([0,\infty])} \mu \int_E d\sigma_h^m(e|h) + \int_E \left(\frac{1}{e} - \mu\right) d\sigma_l^m(e|h)$$

subject to

$$\int_E \left(1 - \frac{1}{e}\right) d\sigma_h^m(e|h) \geq \int_E \left(1 - \frac{1}{e}\right) d\sigma_l^m(e|h) \geq 0$$

$$\frac{\int_{E'} d\sigma_h^m(e|h)}{\int_{E'} d\sigma_l^m(e|l)} \geq \frac{1}{l(\mu)} \text{ for all } E' \subset E \cup \text{supp}(\sigma_l(.|l))$$

Proof. To prove the proposition I will first show that the low type has zero surplus in any $m \in \mathcal{M}_E^r$. The certifier's optimization problem is given by:

$$\max_{m\in M} \mu\rho_h^m + (1-\mu)\rho_l^m$$

such that

$$\int_E [d\sigma_h^m(e|h) - d\sigma_l^m(e|h)] \geq \rho_h^m - \rho_l^m \geq \int_E \frac{1}{e}[d\sigma_h^m(e|h) - d\sigma_l^m(e|h)]$$

$$\int_E d\sigma_h^m(e|h) \geq \rho_h^m \quad \text{and} \quad \int_E \frac{1}{e} d\sigma_l^m(e|h) \geq \rho_l^m$$

$$\frac{\int_{E'} d\sigma_h^m(e|h)}{\int_{E'} d\sigma_l^m(e|l)} \geq \frac{1}{l(\mu)} \text{ for all } E' \subset E$$

For revenue maximization, either low type or high type IR constraint needs to be binding. When the low type's IR constraint is binding the optimization problem reduces to the one in the proposition. This follows from substituting $\rho_l^m = \int_E \frac{1}{e} d\sigma_l^m(e|h)$ and $\rho_h^m = \int_E d\sigma_h^m(e|h)$ - $\int_E \left(1 - \frac{1}{e}\right) d\sigma_l^m(e|h)$.
Thus, to prove the proposition, we first show that the low type never earns rent in a revenue-maximizing menu. Consider for contradiction a revenue-maximizing menu $\chi \in \mathcal{M}$ that offers the low type positive rent, then $\rho_l^\chi = \int_E \frac{1}{e} d\sigma_l^\chi(e|h) - \max\{\int_E \left(\frac{1}{e} - 1\right) d\sigma_h^\chi(e|h), 0\}$ and $\rho_h^\chi = \int_E d\sigma_h^\chi(e|h)$. In particular χ is a solution to

$$\max_{(\sigma_h^m(.|h),\sigma_l^m(.|h))\in\Delta([0,\infty])\times\Delta([0,\infty])} (1-\mu) \int_E \frac{1}{e} d\sigma_l^m(e|h) + \int_E \left(1 - \frac{1-\mu}{e}\right) d\sigma_h^m(e|h)$$

such that

$$\int_E \left(1 - \frac{1}{e}\right) d\sigma_h^m(e|h) \geq \int_E \left(1 - \frac{1}{e}\right) d\sigma_l^m(e|h)$$
$$0 \geq \int_E \left(1 - \frac{1}{e}\right) d\sigma_l^m(e|h)$$
$$\frac{\int_{E'} d\sigma_h^m(e|h)}{\int_{E'} d\sigma_l^m(e|l)} \geq \frac{1}{l(\mu)} \text{ for all } E' \subset E$$

When the low type earns positive rent, i.e. $\int_E \left(\frac{1}{e} - 1\right) d\sigma_h^\chi(e|h) > 0$, it must be that: $\int_E d\sigma_l^\chi(e|h) = 1 - \varepsilon$ for some $\varepsilon > 0$. Moreover, there must exist some $E_l \subset E$ such that $e \in E_l \subset E \cap [0,1]$, E_l is closed and $\int_{E_l} d\sigma_l^\chi(e|h) > 0$. As $\frac{\sigma_h^\chi(e|h)}{\sigma_l^\chi(e|h)} \geq \frac{1}{el(\mu)}$ for all $e \in E$ we have that $\int_{E_l} d\sigma_h^\chi(e|h) > 0$. Then by corollary 2 we get that there must be some $E_h \subset E \cap [0,1]$ such that $\int_{E_h} d\sigma_h^\chi(e|h) > 0$. Fix some arbitrary $e_h \in E_h$ and define $e_l := \min(E_l)$.
Note that $\int_E \frac{1}{e} d\sigma_h^\chi(e|h) = 1$ when the rent to low type is strictly positive, as otherwise, the certifier can increase revenue by slightly increasing the probability of e_h in high type's menu option. In particular if $\int_E \sigma_h^\chi(e|h) < 1$ and $\int_E \frac{1}{e} d\sigma_h^\chi(e|h) > \int_E d\sigma_h^\chi(e|h)$ then $\int_{E^c} d\sigma_h^\chi(e|h) > \int_{E^c} d\sigma_h^\chi(e|l) > 0$; then the certifier can improve his payoff by offering a menu with experiments (σ_h', σ_l'). Where (σ_h', σ_l') is such that $\sigma_l' = \sigma_l^\chi$, and for some $e_h \in E \cap [1, \infty]$ and $\kappa > 0$ small we have $\sigma_h'(e_h|h) = \sigma_h^\chi(e_h|h) + \kappa \min\{e_h \int_{E^c} \sigma_h^\chi(e|l),\ \int_{E^c} \sigma_h^\chi(e|h)\}$, $\sigma_h'(\infty|h) = \sigma_h^\chi(\infty|h) + \kappa \max\{[\int_{E^c} \sigma_h^\chi(e|h) - e_h \int_{E^c} \sigma_h^\chi(e|l)],\ 0\}$, and $\sigma_h'(e|h) = \sigma_h^\chi(e|h)$ for all $e \in E \setminus \{e_h, \infty\}$.

Now to show low type can not earn positive rent construct $\sigma_\theta' : \{l, h\} \to \Delta[0, \infty]$ such that $\sigma_\theta'(e|\theta') = \sigma_\theta^\chi(e|\theta')$ for all θ, θ' and $e \in E \setminus (E_l \cup \{e_h\})$. Moreover, set $\sigma_h'(e_h|h) = \sigma_h^\chi(e_h|h) + \int_{E_l} \frac{\delta}{el(\mu)} d\sigma_l^\chi(e|h)$, $d\sigma_h'(e|h) = d\sigma_h^\chi(e|h) - \frac{\delta}{el(\mu)} d\sigma_l^\chi(e|h)$, $\sigma_l'(e_h|h) = \sigma_l^\chi(e_h|h) + \delta \int_{E_l} d\sigma_l^\chi(e|h)$ and $\sigma_l'(e|h) = (1-\delta)\sigma_l^\chi(e|h)$ whenever $e \in E_l$. Let $\rho_h' = \rho_h^\chi$ and $\rho_l' = \rho_l^\chi + \delta \int_{E_l} \left(\frac{1}{el(\mu)} - 1\right)\left(\frac{1}{e} - \frac{1}{e_h}\right) d\sigma_l^\chi(e|h)$. Define $\chi' := ((\sigma_h', \rho_h'), (\sigma_l', \rho_l'))$. Choose $\delta > 0$ small enough such that $\int_E \left(1 - \frac{1}{e}\right) d\sigma_l'(e|h) \geq \int_E \left(1 - \frac{1}{e}\right) d\sigma_l'(e|h)$ and $0 \geq \int_E \left(1 - \frac{1}{e}\right) d\sigma_l'(e|h)$. By construction, we also get that

$$\text{rev}(\chi') - \text{rev}(\chi) = (1-\mu)\delta \int_{E_l} \left(\frac{1}{el(\mu)} - 1\right)\left(\frac{1}{e} - \frac{1}{e_h}\right) d\sigma_l^\chi(e|h) > 0$$

This concludes the claim that the low type doesn't earn positive rents.
The proof then follows from noting that $\int_E \left(1 - \frac{1}{e}\right) d\sigma_h(e|h) \geq 0 \implies \int_E \left(1 - \frac{1}{e}\right) d\sigma_l(e|h) \geq 0$. Consider for contradiction that $\int_E \left(1 - \frac{1}{e}\right) d\sigma_h(e|h) \geq 0$ and $\int_E \left(1 - \frac{1}{e}\right) d\sigma_l(e|h) < 0$. Then the constraint $\frac{\int_{E'} d\sigma_h(e|h)}{\int_{E'} d\sigma_l(e|l)} \geq \frac{1}{l(\mu)}$ is not biding for some

$E' \subset E$. If instead, the constraint $\frac{d\sigma_h(e|h)}{d\sigma_l(e|l)} \geq \frac{1}{l(\mu)}$ is binding for all $e \in E$ the following calculation gives us a contradiction:

$$\int_E \left(\frac{1}{e} - 1\right) d\sigma_l(e|h) > 0$$

$$\implies \int_E (1-e)\, d\sigma_h(e|h) > 0$$

$$\implies \int_{E\cap\{e>1\}} \left(\frac{1}{e} - 1\right) d\sigma_h(e|h) + \int_{E\cap\{e<1\}} (1-e)\, d\sigma_h(e|h) > 0$$

But

$$\int_E \left(1 - \frac{1}{e}\right) d\sigma_h(e|h) \geq 0$$

$$\implies \int_{E\cap\{e>1\}} \left(\frac{1}{e} - 1\right) d\sigma_h(e|h) + \int_{E\cap\{e<1\}} \left(\frac{1-e}{e}\right) d\sigma_h(e|h) \leq 0$$

$$\implies \int_{E\cap\{e>1\}} \left(\frac{1}{e} - 1\right) d\sigma_h(e|h) + \int_{E\cap\{e<1\}} (1-e)\, d\sigma_h(e|h) \leq 0$$

Finally, the claim follows from the fact that if both the following statements are true then the certifier has a profitable deviation.

$$\exists e' \in E \text{ such that } \frac{d\sigma_h(e'|h)}{d\sigma_l(e'|h)} > \frac{1}{e' l(\mu)}$$

and

$$\int_E \left(1 - \frac{1}{e}\right) d\sigma_l(e|h) < 0$$

The profitable deviation is constructed by considering a menu that provides the same experiment to the high type and at the same cost. But the menu option for low type is changed and priced slightly higher. The low types's new experiment ($\widetilde{\sigma}_l$) has the same distribution for $e \in E \setminus \{e'\}$. But the probability of signal e' in the low type's menu option is increased by shifting mass from the set E^C, until either $\frac{d\widetilde{\sigma}_h(e'|h)}{d\widetilde{\sigma}_l(e'|h)} = \frac{1}{e' l(\mu)}$ or $\int_E \left(1 - \frac{1}{e}\right) d\widetilde{\sigma}_l(e|h) = 0$.

To construct such a deviation, note that $\int_E d\sigma_l(e|h) < \int_E \frac{1}{e} d\sigma_l(e|h) < l(\mu)$, thus we have that $0 < \int_{E^c} \frac{1}{e} d\sigma_l(e|h) < \int_{E^c} d\sigma_l(e|h)$.

Whenever $\int_{E^c} d\sigma_l(e|l) - \frac{1}{e'} \int_{E^c} d\sigma_l(e|h) \geq 0$ then $\widetilde{\sigma}_l$ can be constructed such that for some $\kappa > 0$; $\widetilde{\sigma}_l(e'|h) = \sigma_l(e'|h) + \kappa \int_{E^c} d\sigma_l(e|h)$, $\widetilde{\sigma}_l(0|l) = \sigma(0|l) + \kappa \left[\int_{E^c} d\sigma_l(e|l) - \frac{1}{e'} \int_{E^c} d\sigma_l(e|h)\right]$ and $\widetilde{\sigma}_l(e|h) = \sigma_l(e|h)$ for all $e \notin \{e', 0\}$.

If $\int_{E^c} d\sigma_l(e|l) - \frac{1}{e'}\int_{E^c} d\sigma_l(e|h) < 0$ there exists some $E' \subset E^c$ such that $\int_{E'} d\sigma_l(e|h) > 0$ and $e \in E'$ implies $e > e'$, in this case for choose small enough $\kappa_1, \kappa_2 > 0$ such that $\kappa_1 \int_{E'} d\sigma_l(e|l) + \kappa_2 \int_{E_l} d\sigma_l(e|l) = \frac{1}{e'}\left(\kappa_1 \int_{E'} d\sigma_l(e|h) + \kappa_2 \int_{E_l} d\sigma_l(e|h)\right)$. Define $\widetilde{\sigma}_l(e'|h) = \sigma_l(e'|h) + \kappa_1 \int_{E'} d\sigma_l(e|h) + \kappa_2 \int_{E_l} d\sigma_l(e|h)$, $\widetilde{\sigma}_l(e|h) = (1-\kappa_1)\sigma_l(e|h)$ for all $e \in E'$, $\widetilde{\sigma}_l(e|h) = (1-\kappa_2)\sigma_l'(e|h)$ for all $e \in E_l$ and $\widetilde{\sigma}_l(e|h) = \sigma_l(e|h)$ for all $e \notin \{e'\} \cup E' \cup E_l$. Thus in the optimal menu $\int_E \left(1 - \frac{1}{e}\right) d\sigma_l(e|h) \geq 0$. □

6.5 Proof of Corollary 1

Corollary 1:

If $m \in \mathcal{M}_E^r$ and $\text{rev}(m) > 0$ for some $E \subset [0,\infty]$ then $\int_E d\sigma_h^m(e|h) = 1$.

Proof. Following proposition 1, if $m \in \mathcal{M}_E^r$ then it solves the optimal program mentioned before. In particular, if $\int_E d\sigma_h^m(e|h) < 1$ then implementablity requires $\int_E \frac{1}{e} d\sigma_h^m(e|h) \leq \int_E d\sigma_h^m(e|h) \leq 1 - \varepsilon$ for some $\varepsilon > 0$. Thus there exist $E_h, E_l \subset E^C$ such that $0 < \int_{E_h} d\sigma_h^m(e|h), \int_{E_l} d\sigma_h^m(e|l) \leq \varepsilon$. By lemma 5 we get that there exists $e' > 1 \in E$ (if $\infty \in E$ then let $e' = \infty$). If $\int_{E_h \cap E_l} \sigma_h^m(e|h), \int_{E_h \cap E_l} \sigma_h^m(e|l) > 0$ then the certifier has a profitable deviation thus $m \notin \mathcal{M}_E^r$. To construct a profitable deviation of the certifier, we consider a menu m' such that the low type's payment and experiment are equal to the low type's payment and experiment under the menu m. For the high type if $\frac{1}{e'}\int_{E_h \cap E_l} \sigma_h^m(e|h) \leq \int_{E_h \cap E_l} \sigma_h^m(e|l)$ let $\sigma_h^{m'}(e|h) := \sigma_h^m(e|h)$ whenever $e \in E \setminus \{e'\}$, $\sigma_h^{m'}(e'|h) := \sigma_h^m(e'|h) + \int_{E_h \cap E_l} \sigma_h^m(e|h)$ and $\rho_h^{m'} := \rho_h^m + \int_{E_h \cap E_l} \sigma_h^m(e|h)$. When $\frac{1}{e'}\int_{E_h \cap E_l} \sigma_h^m(e|h) > \int_{E_h \cap E_l} \sigma_h^m(e|l)$, let $\sigma_h^{m'}(e|h) := \sigma_h^m(e|h)$ whenever $e \in E \setminus \{e', \infty\}$, $\sigma_h^{m'}(e'|h) := \sigma_h^m(e'|h) + e' \int_{E_h \cap E_l} \sigma_h^m(e|l)$, $\sigma_h^{m'}(\infty|h) := \sigma_h^m(\infty|h) + \left[\int_{E_h \cap E_l} \sigma_h^m(e|h) - e' \int_{E_h \cap E_l} \sigma_h^m(e|l)\right]$, and $\rho_h^{m'} := \rho_h^m + e' \int_{E_h \cap E_l} \sigma_h^m(e|h)$. If instead $\int_{E_h \cap E_l} \sigma_h^m(e|h) = 0$, then $\sigma_h^m(\infty|h), \sigma_h^m(0|l) > 0$. Choose $\delta > 0$ small enough such that $\sigma_h^m(\infty|h) > \delta$, $\sigma_h^m(0|l) > \frac{\delta}{e'}$. The certifier has a profitable deviation m' such that $\sigma_l^{m'} := \sigma_l^m$, $\rho_l^{m'} := \rho_l^m$, $\sigma_h^{m'}(e|h) := \sigma_h^m(e|h)$ whenever $e \in E \setminus \{e'\}$, $\sigma_h^{m'}(e'|h) := \sigma_h^m(e'|h) + \delta$ and $\rho_h^{m'} = \rho_h^m + \delta$. The menu $m' \in \mathcal{M}_E$ for all the cases above by construction. Thus the corollary follows from contradiction. □

6.6 Separating Equilibrium

Corollary 2: If an equilibrium $((E_m)_{m \in \mathcal{M}}), m^*) \in \mathcal{E}_r$ is separating then $e^* := \inf(E_{m^*}) \geq \frac{1}{\mu}$. Moreover, if for an equilibrium $((E_m)_{m \in \mathcal{M}}), m^*) \in \mathcal{E}_r$ it holds that $e^* := \inf(E_{m^*}) > \frac{1}{\mu}$, then the equilibrium is separating.

Proof. Sufficiency: Let $((E_m)_{m\in\mathcal{M}}), m^*) \in \mathcal{E}_r$. If E_{m^*} is such that $e \in E_{m^*}$ implies that $e \geq 1$ then by applying proposition 1 we get that m^* needs to be a solution to

$$\max_{(\sigma_h^m(.|h),\sigma_l^m(.|h))\in\Delta([0,\infty])\times\Delta([0,\infty])} \mu \int_{E_{m^*}} d\sigma_h^m(e|h) + \int_{E_{m^*}} \left(\frac{1}{e} - \mu\right) d\sigma_l^m(e|h)$$

subject to

$$\int_{E_{m^*}} \left(1 - \frac{1}{e}\right) d\sigma_h^m(e|h) \geq \int_{E_{m^*}} \left(1 - \frac{1}{e}\right) d\sigma_l^m(e|h) \geq 0$$

$$d\sigma_h^m(e|h) \geq \frac{1}{el(\mu)} d\sigma_l^m(e|h) \text{ for all } e \in E_{m^*}$$

The non-negativity constraint in the IC conditions is irrelevant as $e > 1$ for all $e \in E_{m^*}$. Now notice that if $e^* \geq \frac{1}{\mu}$ then the revenue of the certifier is decreasing in $d\sigma_l^m(e|h)$ for all $e \in E_{m^*}$. This means that the low type's menu option assigns zero probability to the set E_{m^*}. In particular, the revenue maximizing menu m^* is such that $\int_{E_{m^*}} d\sigma_h^{m^*}(e|h) = 1$, $\rho_h^{m^*} = 1$, $\sigma_l^{m^*} = \Phi$, $\rho_l^{m^*} = 0$. This menu is separating. Notice when $e^* > \frac{1}{\mu}$ then the revenue is strictly decreasing in $\sigma_l^{m^*}(e|h)$ for all $e \in E_{m^*}$ thus the menu with $\int_{E_{m^*}} d\sigma_h^{m^*}(e|h) = 1$, $\rho_h^{m^*} = 1$, $\sigma_l^{m^*} = \Phi$, $\rho_l^{m^*} = 0$ is uniquely optimal.

For necessity, let $((E_m)_{m\in\mathcal{M}}), m^*) \in \mathcal{E}_r$ be a separating equilibrium. I will show that if there exists $e \in E_{m^*}$ such that $e < \frac{1}{\mu}$ then $m^* \notin \mathcal{M}^r_{E_{m^*}}$.

Case I: There exists $0 < e_l < 1 \in E_{m^*}$. As m^* is separating so by lemma 5 we get that there exists some $e_h > 1 \in E_{m^*}$. Now consider the following menu: $\chi = ((\sigma_\theta, \rho_\theta))_{\theta\in\{l,h\}}$. Where $\sigma_h^\chi(e_h|h) = \frac{e_h(1-e_l)}{e_h-e_l}$, $\sigma_h^\chi(e_l|h) = \frac{e_l(e_h-1)}{e_h-e_l}$, $\rho_h = 1$ and $\sigma_l^\chi(e_h|h) = l(\mu)e_l\frac{e_h(1-e_l)}{e_h-e_l}$, $\sigma_l^\chi(e_l|h) = l(\mu)e_l\frac{e_l(e_h-1)}{e_h-e_l}$, $\rho_l = e_l l(\mu)$. Observe that $\chi \in \mathcal{M}_{E_{m^*}}$, and $\text{rev}(\chi) = \mu + (1-\mu)l(\mu)e_l$. The claim follows from noting that the revenues from a separating menu is bounded above by μ.

Case II: $e \in E$ implies $e > 1$ and there exists $e_h \in E$ such that $\frac{1}{\mu} > e_h > 1$. If $e_h \geq \frac{1}{l(\mu)}$, the certifier can improve revenue by offering the same menu option to both types. This is $\sigma_h = \sigma_l$ where $\sigma_h(e_h|h) = 1$, $\sigma_h(e_h|l) = \frac{1}{e_h}$ and $\rho_h = \rho_l = \frac{1}{e_h}$.

If $e_h < \frac{1}{l(\mu)}$. The certifier can improve the revenue by offering a menu such that $\sigma_h(e_h|h) = 1$, $\sigma_l(e_h|h) = e_h l(\mu)$ and $\rho_h = 1 - l(\mu)(e_h - 1)$, $\rho_l = l(\mu)$. □

6.7 Proof of Theorem 1

Theorem 1: If $((E_m)_{m\in\mathcal{M}}, m^*) \in \mathcal{E}_r$ such that E_{m^*} is countable, then there exists an outcome equivalent equilibrium $((E'_m)_{m\in\mathcal{M}}, m') \in \mathcal{E}_r$ such that $m' \in \mathbf{cvx}(\mathcal{T}(E_{m^*}))$.

Combining corollary 1, the constraint $\int_E \left(1-\frac{1}{e}\right) d\sigma_\theta(e|h) \geq 0$ and remark 2. We get that it suffices to consider menus such that $\text{supp}(\sigma_\theta(.|l)) \cap E^C \subset \{0,1\}$. With this in mind, I restrict the space of choices from $\Delta([0,\infty]) \times \Delta([0,\infty])$ to Λ, where

$$\Lambda := \{(\eta_h, \eta_l) \in \Delta([0,\infty]) \times \Delta([0,\infty]) |\, \text{supp}(\eta_\theta) \cap E^C = \{1\}\}$$

Λ is a closed and convex set[16]

For each $E \subset [0,\infty]$. The set of **feasible**[17] signal distributions wrt E is given by $\mathcal{K}_E \subset \Lambda$. Where $\mathcal{K}_E$ is the set of signal distributions $(\eta_h, \eta_l) \in \Lambda$ such that

$$\int_E \left(1-\frac{1}{e}\right) d\eta_h(e) \geq \int_E \left(1-\frac{1}{e}\right) d\eta_l(e)$$

$$\int_E \left(1-\frac{1}{e}\right) d\eta_l(e) \geq 0$$

$$\int_E d\eta_h(e) = 1$$

and

$$\int_{E'} d\eta_h(e) \geq \int_{E'} \frac{1}{el(\mu)} d\eta_l(e) \ \text{ for all } E' \subset E$$

The first claim establishes the existence, in terms of restrictions on E, of a maximizer of some linear functional over $\mathcal{K}_E$.

Claim 1. *For any $\varepsilon > 0$ and $E \subset [\varepsilon, \infty] \setminus \{1\}$ closed, the set $\mathcal{K}_E$ is compact and convex.*

Proof. $[0,\infty]$ is a compact Polish space, thus $\Delta([0,\infty])$ is compact and Polish. Using the topology of weak convergence on $\Delta([0,\infty])$ and appropriately defined product topology on $\Delta([0,\infty]) \times \Delta([0,\infty])$.[18] If $(\eta_h^n, \eta_l^n) \to (\eta_h, \eta_l)$in the product topology then $\eta_h^n \to_w$

[16] The topology here is product topology, where the topology on each component is the topology of weak convergence.

[17] The requirement of feasibility wrt E is stronger than obedience wrt E as feasibility builds in revenue-maximizing properties that are discussed in proposition 1.

[18] Product of compact and polish spaces is compact and polish. A closed subset of such a space is also compact and polish. Moreover, the product of locally convex spaces is also locally convex.

η_h and $\eta_l^n \to_w \eta_l$ in $\Delta([0,\infty])$. Noting that $1-\frac{1}{e}$ and $\frac{1}{el(\mu)}$ are continuous and bounded on $[\varepsilon,\infty]$.[19] Thus weak convergence of measures η_h^n and η_l^n give us convergence of following integrals

$$\lim_{n\to\infty}\int_E \left(1-\frac{1}{e}\right) d\eta_\theta^n(e) = \int_E \left(1-\frac{1}{e}\right) d\eta_\theta(e)$$

and,

$$\lim_{n\to\infty}\left[\int_{E'} d\eta_h^n(e) - \int_{E'} \frac{1}{el(\mu)} d\eta_l^n(e)\right] = \int_{E'} d\eta_h(e) - \int_{E'} \frac{1}{el(\mu)} d\eta_l(e) \text{ for all } E' \subset E$$

As $(\eta_h^n,\eta_l^n) \in \mathcal{K}_E$, the above implies that $(\eta_h,\eta_l) \in \mathcal{K}_E$. Thus $\mathcal{K}_E$ is closed and hence compact. The convexity follows from noting that the $\mathcal{K}_E$ is defined by elements of $\Delta([0,\infty]) \times \Delta([0,\infty])$ that satisfy certain linear inequalities. Thus any convex combination of elements in $\mathcal{K}_E$ also satisfies these linear inequalities. This proves the claim. □

A consequence of the claim and Krein–Milman theorem is that the set of extreme points $ex(\mathcal{K}_E)$ is nonempty. As a convention if some functional f doesn't attain its maximum in the set $\mathcal{K}_E$, then $\arg\max_{\mathcal{K}_E}\{f(\eta_h,\eta_l)\} = \varnothing$. For all the following statements in the proof, I assume that E is countable. In particular, the support of all the relevant distributions is discreet.[20]

Let $\mathcal{T}(\mathcal{K}_E) := \{(\eta_h,\eta_l) \in \mathcal{K}_E \mid |\operatorname{supp}(\eta_h)\cap E| \le 3,\ |\operatorname{supp}(\eta_l)\cap E| \le 2\}$.

Claim 2. *If* $E\cap[0,1]=\varnothing$ *then* $\arg\max_{\mathcal{K}_E}\{\mu\int_E d\eta_h(e) + \int_E \left(\frac{1}{e}-\mu\right) d\eta_l(e)\} \subset \mathcal{T}(\mathcal{K}_E)$.

Proof. Let $(\eta_h^*,\eta_l^*) \in \arg\max_{\mathcal{K}_E}\{\mu\int_E d\eta_h(e) + \int_E \left(\frac{1}{e}-\mu\right) d\eta_l(e)\}$. By solving the optimization problem in proposition 1 (see proof of corollary 2 for example), we get that $|\operatorname{supp}(\eta_h^*)\cap E| > 1$ only if $\eta_l^*(1) = 1$, in which case we can write $(\eta_h^*,\eta_l^*) = \sum_{e\in E}\eta_h(e)(\eta_h^e,\eta_l^e)$. Where $\eta_l^e = \eta_l^*$ and $\eta_h^e(e) = 1$ for all $e\in E$. □

In particular if $(\eta_h',\eta_l') \in \arg\max_{\mathcal{K}_E}\{\mu\int_E d\eta_h(e) + \int_E \left(\frac{1}{e}-\mu\right) d\eta_l(e)\}$ are such that $\operatorname{supp}(\eta_h)\cap[0,1]\cap E = \varnothing$ then $(\eta_h',\eta_l') \in \arg\max_{\mathcal{K}_{E\setminus[0,1]}}\{\mu\int_E d\eta_h(e) + \int_E \left(\frac{1}{e}-\mu\right) d\eta_l(e)\}$.

[19]This requires E to be bounded away from 0

[20]The proofs can be extended to continuous distribution, the arguments presented rely on discreet support distributions mostly for exposition reasons.

Claim 3. *For any* $(\eta_h, \eta_l) \in \arg\max_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$*. If* $\text{supp}(\eta_h) \cap E \cap [0,1] \neq \varnothing$ *then* $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$*.*

Proof. Consider for contradiction $\text{supp}(\eta_h) \cap E \cap [0,1] \neq \varnothing$ and $\text{supp}(\eta_l) \cap E \cap [0,1] = \varnothing$. Let $E_l \subset E \cap [0,1]$ be such that $\int_{E_l} d\eta_h(e) > 0$, fix some $e_l \in E_l$. By assumption $\int_{E_l} d\eta_l(e) = 0$. Let $E_h \subset E \cap [1,\infty]$ be such that $\int_{E_h} d\eta_l(e) > 0$, fix some $e_h \in E_h$.

First, consider the case when $\eta_l(1) = 0$, then we get that $\int_E d\eta_h(e) = \int_E d\eta_l(e) = 1$ and $\int_E \frac{1}{e} d\eta_l \geq \int_E \frac{1}{e} d\eta_h$. As $\frac{d\eta_l(e)}{d\eta_h(e)} \leq el(\mu)$ for all $e \in E$, we get that $\int_E \frac{1}{e} d\eta_l(e) \leq l(\mu)$. Now construct η'_h, η'_l such that $\eta'_h = \eta_h$ and $\eta'_l(e) = \eta_l(e)$ for all $e \notin \{e_l,\ e_h\}$. Let $\eta'_l(e_h) = \eta_l(e_h) - \varepsilon$, $\eta'_l(e_l) = \eta_l(e_l) + \varepsilon$, and $1 - \int_E \frac{1}{e} d\eta'_l(e) = 1 - \int_E \frac{1}{e} d\eta_l(e) - \varepsilon\left(\frac{1}{e_l} - \frac{1}{e_h}\right)$. For small enough $\varepsilon > 0$ we get that $(\eta'_h, \eta'_l) \in \mathcal{K}_E$ and

$$\mu \int_E d\eta'_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta'_l(e) - \mu \int_E d\eta_h(e) - \int_E \left(\frac{1}{e} - \mu\right) d\eta_l(e)$$
$$= \mu\varepsilon\left(\frac{1}{e_l} - \frac{1}{e_h}\right) > 0$$

This is a contradiction, thus it must be that $\int_{E_l} d\eta_l(e) > 0$.

Now, consider the case when $\eta_l(1) > 0$. Construct η'_h, η'_l such that for all $e \notin \{e_h, 1\} \cup E_l$ we have $\eta'_\theta(e) = \eta_\theta(e)$. Let $\eta'_h(e_h) = \eta_h(e_h) + \varepsilon \int_{E_l} d\eta_h(e)$, $\eta'_l(e_h) = \eta_l(e_h) + \varepsilon e_h l(\mu) \int_{E_l} d\eta_h(e)$, $\eta'_h(e) = (1-\varepsilon)\eta_h(e)$ for all $e \in E_l$, $\eta'_l(e) = \eta_l(e)$ for all $e \in E_l \setminus \{e_l\}$, $\eta'_l(e_l) = \eta_l(e_l) + \varepsilon l(\mu)\frac{(e_h-1)e_l}{1-e_l} \int_{E_l} d\eta_h(e)$, $\eta'_h(1) = \eta_h(1)$, $1 - \int_{E\cup\{1\}} \frac{1}{e} d\eta'_h(e) = 1 - \int_{E\cup\{1\}} \frac{1}{e} d\eta_h(e) + \varepsilon \int_{E_l} \left(\frac{1}{e} - \frac{1}{e_h}\right) d\eta_h(e)$, $1 - \int_{E\cup\{1\}} \frac{1}{e} d\eta'_l(e) = 1 - \int_{E\cup\{1\}} \frac{1}{e} d\eta_l(e)$, and $\eta'_l(1) = \eta_l(1) - \varepsilon l(\mu)\frac{e_h - e_l}{1-e_l} \int_{E_l} d\eta_h(e)$. For small enough $\varepsilon > 0$ we get that $(\eta'_h, \eta'_l) \in \mathcal{K}_E$ and

$$\mu \int_E d\eta'_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta'_l(e) - \mu \int_E d\eta_h(e) - \int_E \left(\frac{1}{e} - \mu\right) d\eta_l(e)$$
$$= (1-\mu)\varepsilon l(\mu)\frac{e_h - e_l}{1-e_l} \int_{E_l} d\eta_h(e) > 0$$

This is a contradiction, thus it must be that $\int_{E_l} d\eta_l(e) > 0$. □

For the rest of the claims let $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$.

Partition $\mathcal{K}_E$ into K_0^E and K_1^E. Where $K_0^E := \{(\eta_h, \eta_l) \in \mathcal{K}_E | \int_E \left(1 - \frac{1}{e}\right) d\eta_l(e) = 0\}$, and $K_1^E = \mathcal{K}_E \setminus K_1^E$.

Claim 4. *For any* $(\eta_h, \eta_l) \in \arg\max_{\mathcal{K}_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$*. If* $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$ *and* $\int_E \left(1 - \frac{1}{e}\right) d\eta_l(e) > 0$*, then* $\text{supp}(\eta_l) \cap E \cap (\frac{1}{l(\mu)}, \infty] = \varnothing$ *or* $|\text{supp}(\eta_l) \cap E \cap [1,\infty]| = 1$*.*

Proof. Note that $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$ implies that there is some $e_l \in E \cap [0,1]$ such that $\eta_h(e_l) > \eta_l(e_l) > 0$. By assumption we have $\int_E \left(1 - \frac{1}{e}\right) d\eta_l(e) > 0$. As $\int_E \left(1 - \frac{1}{e}\right)(d\eta_h(e) - d\eta_l(e)) \geq 0$, we get that $\int_E \left(1 - \frac{1}{e}\right) d\eta_h(e) > 0$. For contradiction let there be some $e', e_h \in E \cap [1, \infty]$ such that $e' < e_h$, $\eta_l(e_h) > 0$, and $\frac{1}{l(\mu)} < e_h$.

Now construct η'_h, η'_l such that $\eta'_\theta(e) = \eta_\theta(e)$ for $e \notin \{e_h, e', 0, 1\}$. Let $\eta'_h(e_h) = \eta_h(e_h) - \varepsilon$, $\eta'_h(e') = \eta_h(e') + \varepsilon$, $1 - \int_E d\eta'_h(e) = 1 - \int_E d\eta_h(e) - \left(\frac{1}{e'} - \frac{1}{e_h}\right)\varepsilon$, $\eta'_l(e_h) = \eta_l(e_h) - e_h l(\mu)\varepsilon$, $\eta'_l(e') = \eta_l(e') + e' l(\mu)\varepsilon$, $\eta'_l(1) = \eta_l(1) + \varepsilon l(\mu)(e_h - e')$, and $1 - \int_E d\eta'_l(e) = 1 - \int_E d\eta_l(e) - l(\mu)\varepsilon(e_h - e')$. For small enough $\varepsilon > 0$, we get $(\eta'_h, \eta'_l) \in \mathcal{K}_E$ as

$$\left(1 - \frac{1}{e_h}\right)\varepsilon(e_h l(\mu) - 1) - \left(1 - \frac{1}{e'}\right)\varepsilon(l(\mu)e' - 1)$$

$$= \varepsilon(e_h - e')\left(l(\mu) - \frac{1}{e_h e'}\right) > 0$$

Finally, note that

$$\mu \int_E d\eta'_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta'_l(e) - \mu \int_E d\eta^*_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta^*_l(e)$$

$$= \mu\varepsilon l(\mu)(e_h - e') > 0$$

□

A direct consequence of claim 4 is that for any $(\eta_h, \eta_l) \in \arg\max_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$, if $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$, then $\eta_l(1) > 0$.

Claim 5. *For any* $(\eta_h, \eta_l) \in \arg\max_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$. *Such that* $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$. *If* $|\text{supp}(\eta_h) \cap E \cap [1,\infty]| \geq 2$ *then for any* $e', e_h \in \text{supp}(\eta_h) \cap E \cap [1, \infty]$, $e' < e_h$ *implies that* $\frac{\eta_l(e_h)}{\eta_h(e_h)} = 0$ *or* $e_h l(\mu)$, *and if* $\frac{\eta_l(e_h)}{\eta_h(e_l)} = e_h l(\mu)$ *then* $\frac{\eta_l(e')}{\eta_h(e')} = e' l(\mu)$.

Proof. Note that $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$ implies that there is some $e_l \in E \cap [0,1]$ such that $\eta_h(e_l) > \eta_l(e_l) > 0$. In particular, by claim 4 we have $\eta_l(1) > 0$.

First, for contradiction assume that $0 < \frac{\eta_l(e_h)}{\eta_h(e_h)} < e_h l(\mu)$. Construct (η'_h, η'_l) such that for all $e \notin \{1, 0, e_h, e_l, e'\}$ we have $\eta'_\theta(e) = \eta_\theta(e)$. Let $\eta'_h(e_h) = \eta_h(e_h) - \varepsilon$, $\eta'_h(e_l) = \eta_h(e_l) - \delta$, $\eta'_h(e') = \eta_h(e') + \varepsilon + \delta$, $\eta'_l(e_h) = \eta_l(e_h) - \varepsilon\psi$, $\eta'_l(e_l) = \eta_l(e_l) - e_l l(\mu)\delta$, $\eta'_l(e') = \eta_l(e') + \varepsilon\psi + e_l l(\mu)\delta + \kappa$, and $\eta'_l(1) = \eta_l(1) - \kappa$. Where $\kappa = \frac{1}{e'-1}(\varepsilon\psi\left(1 - \frac{e'}{e_h}\right) - \delta l(\mu)(e' - e_l))$,

$\psi > \frac{\eta_l(e_h)}{\eta_h(e_h)}$, $\varepsilon > \delta \frac{e'-e_l}{e_h-e'} \frac{l(\mu)e_h}{\psi}$ and $\delta > 0$. By choosing ψ, ε, and δ close enough to the lower bounds mentioned respectively we get that $(\eta_h', \eta_l') \in \mathcal{K}_E$, by construction we see that

$$\mu \int_E d\eta_h'(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta_l'(e) - \mu \int_E d\eta_h(e) - \int_E \left(\frac{1}{e} - \mu\right) d\eta_l(e)$$

$$= (1-\mu)\kappa > 0$$

Now consider $e' < e_h$ where $\frac{\eta_l(e')}{\eta_h(e')} \neq e' l(\mu)$. Construct η_h', η_l' such that $\eta_h'(e) = \eta_h(e)$ for all $e \in [0, \infty]$. For $e \notin \{e_h, e', e_l, 1\}$ let $\eta_l'(e) = \eta_l(e)$. Let $\eta_l'(e_h) = \eta_l(e_h) - \varepsilon$, $\eta_l'(e_l) = \eta_l(e_l) - \delta$, $\eta_l'(1) = \eta_l(1) - \kappa$ and $\eta_l'(e') = \eta_l(e') + \varepsilon + \delta + \kappa$. Where $\kappa = \frac{e'}{e'-1}\left(\varepsilon\left(\frac{1}{e'} - \frac{1}{e_h}\right) - \delta\left(\frac{1}{e_l} - \frac{1}{e'}\right)\right)$, $\varepsilon > \delta \frac{(e'-e_l)e_h}{(e_h-e')e_l}$ and $\delta > 0$. By choosing ε and δ close enough to the lower bounds mentioned respectively we get that $(\eta_h', \eta_l') \in \mathcal{K}_E$, by construction we see that

$$\mu \int_E d\eta_h'(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta_l'(e) - \mu \int_E d\eta_h(e) - \int_E \left(\frac{1}{e} - \mu\right) d\eta_l(e)$$

$$= (1-\mu)\kappa > 0$$

□

Following claim 5, I proceed by dividing the proof into two cases. First, I consider $(\eta_h, \eta_l) \in \arg\max_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$, such that $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$ and $\frac{\eta_l(e)}{\eta_h(e)} = el(\mu)$ for all $e \in E \cap [1, \infty]$. Second, I consider the case $(\eta_h, \eta_l) \in \arg\max_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$, such that $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$ and $\frac{\eta_l(e)}{\eta_h(e)} < el(\mu)$ for some $e \in E \cap [1, \infty]$. Before proceeding I will first prove some claims that simplify the optimization problem.

Claim 6. *For any* $(\eta_h, \eta_l) \in \arg\max_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$, *such that* $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$. *If* $\int_E \left(1 - \frac{1}{e}\right) d\eta_l(e) = 0$ *then* $\int_E \left(1 - \frac{1}{e}\right) d\eta_l(e) = \int_E \left(1 - \frac{1}{e}\right) d\eta_l(e)$.

Proof. Fix some $(\eta_h, \eta_l) \in \arg\max_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$ such that $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$. The proof follows in two steps:

The first argument is similar to the last part of the proof of proposition 1. We want to show that if $\int_E \left(1 - \frac{1}{e}\right) d\eta_l(e) = 0$ then there exist some $e_h \in \text{supp}(\eta_h) \cap E \cap [1, \infty]$ such that $\frac{\eta_l(e_h)}{\eta_h(e_h)} < e_h l(\mu)$. Whenever $\int_E \left(1 - \frac{1}{e}\right) d\eta_l(e) = 0$ we have that there exists

$E_l \subset \text{supp}(\eta_l) \cap E \cap [0,1]$ such that $\int_{E_l} d\eta_l(e) > 0$. Assume for contradiction that for all $e \in \text{supp}(\eta_h) \cap E \cap [1,\infty]$ we have $\frac{\eta_l(e_h)}{\eta_h(e_h)} = el(\mu)$. Then it follows that

$$l(\mu) \int_{E\cap[1,\infty]} (e-1) d\eta_h(e) = \int_{E\cap[0,1]} \left(\frac{1}{e} - 1\right) d\eta_l(e) \le l(\mu) \int_{E\cap[0,1]} (1-e) d\eta_h(e)$$

Feasibility further requires that

$$\int_{E\cap[0,1]} \frac{1}{e} d\eta_h(e) + \int_{E\cap[1,\infty]} \frac{1}{e} d\eta_h(e) \le 1$$

$$\implies \int_{E\cap[0,1]} \left(\frac{1}{e} - 1\right) d\eta_h(e) \le \int_{E\cap[1,\infty]} \left(1 - \frac{1}{e}\right) d\eta_h(e)$$

Now note that $\int_{E\cap[0,1]} \left(\frac{1}{e} - 1\right) d\eta_h(e) > \int_{E\cap[0,1]} (1-e) d\eta_h(e)$ and $\int_{E\cap[1,\infty]} \left(1 - \frac{1}{e}\right) d\eta_h(e) < \int_{E\cap[1,\infty]} (e-1) d\eta_h(e)$. Thus, we have a contradiction.

For the second argument, we want to show that if $0 = \int_E \left(1 - \frac{1}{e}\right) \eta_l < \int_E \left(1 - \frac{1}{e}\right) \eta_h$ then for any $E_h \subset \text{supp}(\eta_h) \cap E \cap [1,\infty]$, we have $\int_{E_h} d\eta_l(e) = l(\mu) \int_{E_h} ed\eta_h(e)$. Assume for contradiction that there is some $E_h \subset [1,\infty] \cap E$ such that $\int_{E_h} d\eta_h(e) > 0$ and $\int_{E_h} d\eta_l(e) < l(\mu) \int_{E_h} ed\eta_h(e)$, in particular E_h can be chosen such that $\frac{d\eta_l(e)}{d\eta_h(e)} < el(\mu)$ for all $e \in E_h$. Also, note that there is some $e_l \in E \cap [0,1]$.

Construct η'_h, η'_l such that for all $e \notin E_h \cup \{e_l, 1, 0\}$ we have $\eta'_\theta(e) = \eta_\theta(e)$. Define $\eta'_h(e) = (1-\varepsilon)\eta_h(e)$ for all $e \in E_h$, $\eta'_h(e_l) = \eta_h(e_l) + \varepsilon \int_{E_h} d\eta_h(e)$, $1 - \int_E d\eta'_h(e) = 1 - \int_E d\eta_h(e) - \varepsilon \int_{E_h} \left(\frac{1}{e_l} - \frac{1}{e}\right) d\eta_h(e)$, $\eta'_l(e) = \left(1 + \varepsilon l(\mu) \frac{e(1-e_l)}{e-1} \frac{\eta_h(e)}{\eta_l(e)}\right) \eta_l(e)$ for all $e \in E_h$, $\eta'_l(e_l) = \eta_l(e_l) + \varepsilon l(\mu) e_l \int_{E_h} d\eta_h(e)$, $\eta'_l(1) = \eta_l(1) - \varepsilon l(\mu) \int_{E_h} \frac{e-e_l}{e-1} d\eta_h(e)$. By choosing $\varepsilon > 0$ small enough, we get that $(\eta'_h, \eta'_l) \in \mathcal{K}_E$, and

$$\mu \int_E d\eta'_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta'_l(e) - \mu \int_E d\eta_h(e) - \int_E \left(\frac{1}{e} - \mu\right) d\eta_l(e)$$

$$= (1-\mu)\varepsilon l(\mu) \int_{E_h} \frac{e - e_l}{e-1} d\eta_h(e) > 0$$

□

A consequence of claim 6 is that, if $(\eta_h, \eta_l) \in \arg\max_{K_0^E} \{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$ and $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$ then by there is some $e_h \in \text{supp}(\eta_h) \cap E \cap [1,\infty]$ such that $\frac{\eta_l(e_h)}{\eta_h(e_h)} < e_h l(\mu)$.

Claim 7. *For any* $(\eta_h, \eta_l) \in \arg\max_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$. *Such that* $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$. *If* $\int_E \left(1 - \frac{1}{e}\right)\eta_l < \int_E \left(1 - \frac{1}{e}\right)\eta_h$ *then for any* $e_h \in \text{supp}(\eta_h) \cap E \cap [1, \infty]$, $\frac{\eta_l(e_h)}{\eta_h(e_h)} = e_h l(\mu)$.

Proof. Note that $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$ implies that there is some $e_l \in E \cap [0,1]$ such that $\eta_h(e_l) > \eta_l(e_l) > 0$. In particular, $\eta_l(1) > 0$. For contradiction let $\frac{\eta_l(e_h)}{\eta_h(e_l)} < el(\mu)$. Construct η'_h, η'_l such that for all $e \notin \{e_h, e_l, 1, 0\}$ we have $\eta'_\theta(e) = \eta_\theta(e)$. Define $\eta'_h(e_h) = \eta_h(e_h) - \varepsilon$, $\eta'_h(e_l) = \eta_h(e_l) + \varepsilon$, $1 - \int_E d\eta'_h(e) = 1 - \int_E d\eta_h(e) - \delta$, $\eta'_l(e_h) = \eta_l(e_h) + p$, $\eta'_l(e_l) = \eta_l(e_l) + \kappa - p$, $\eta'_l(1) = \eta_l(1) - \kappa$. Where $\kappa = p\frac{e_h - e_l}{(1-e_l)e_h}$, $p = \varepsilon e_l l(\mu)\frac{(1-e_l)e_h}{e_h - 1}$, $\delta = \varepsilon\frac{e_h - e_l}{e_h e_l}$, $\varepsilon > 0$. By choosing ε close enough to 0 we get that $(\eta'_h, \eta'_l) \in \mathcal{K}_E$, and

$$\mu \int_E d\eta'_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta'_l(e) - \mu \int_E d\eta_h(e) - \int_E \left(\frac{1}{e} - \mu\right) d\eta_l(e)$$
$$= (1-\mu)\kappa > 0$$

□

In particular for $(\eta_h, \eta_l) \in \arg\max_{K_1^E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$, such that $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$ and $\frac{\eta_l(e)}{\eta_h(e)} < el(\mu)$ for some $e \in E \cap [1, \infty]$, it must be that $\int_E \left(1 - \frac{1}{e}\right)\eta_l = \int_E \left(1 - \frac{1}{e}\right)\eta_h$.

Claim 8. *For any* $(\eta_h, \eta_l) \in \arg\max_{K_1^E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$. *Whenever* $e_l \in \text{supp}(\eta_h) \cap E \cap [0,1]$ *then* $\eta_l(e_l) = e_l l(\mu)$.

Proof. Assume for contradiction there is some $e_l < 1$ such that $\eta_l(e_l) < e_l l(\mu)\eta_h(e_l)$. As $(\eta_h, \eta_l) \in K_1^E$ we get that $1 - \int_{E\cup\{1\}} \frac{1}{e} d\eta_l(e) > 0$. Construct η'_h, η'_l such that $\eta'_h = \eta_h$ and $\eta'_l(e) = \eta_l(e)$ for all $e \notin \{e_h, e_l\}$. Let $\eta'_l(e_h) = \eta_l(e_h) - \varepsilon$, $\eta'_l(e_l) = \eta_l(e_l) + \varepsilon$, $1 - \int_{E\cup\{1\}} \frac{1}{e} d\eta'_l(e) = 1 - \int_{E\cup\{1\}} \frac{1}{e} d\eta_l(e) - \varepsilon\left(\frac{1}{e_l} - \frac{1}{e_h}\right)$. By choosing $\varepsilon > 0$ small enough we get that $(\eta'_h, \eta'_l) \in \mathcal{K}_1^E$ and

$$\mu \int_E d\eta'_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta'_l(e) - \mu \int_E d\eta_h(e) - \int_E \left(\frac{1}{e} - \mu\right) d\eta_l(e)$$
$$= (1-\mu)\varepsilon\left(\frac{1}{e_l} - \frac{1}{e_h}\right) > 0$$

Thus the claim follows from contradiction. □

Claim 9. *For any* $(\eta_h, \eta_l) \in \arg\max_{K_0^E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$. *If* $|\operatorname{supp}(\eta_l) \cap E \cap [0,1]| > 1$, *then* $\frac{\eta_l(e)}{\eta_h(e)} = l(\mu)e$ *for all* $e \in \operatorname{supp}(\eta_h) \cap E \cap [0,1]$.

Proof. If $(\eta_h, \eta_l) \in \arg\max_{K_0^E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$ then by claim 6 there is some $e_h \in \operatorname{supp}(\eta_h) \cap E \cap [1, \infty]$ such that $\frac{\eta_l(e_h)}{\eta_h(e_h)} < e_h l(\mu)$. Let $e' < e_l \in \operatorname{supp}(\eta_h) \cap E \cap [0,1]$, for contradiction assume that

Case I: $\eta_l(e') < e' l(\mu) \eta_h(e')$. Construct (η'_h, η'_l) such that $\eta'_\theta(e) = \eta_\theta(e)$ for all $e \notin \{e_h, e_l, e', 1, 0\}$. Let $\eta'_h(e_h) = \eta_h(e_h) - \varepsilon$, $\eta'_h(e') = \eta_h(e') - \delta$, $\eta'_h(e_l) = \eta_h(e_l) + \varepsilon + \delta$, $\eta'_l(e_h) = \eta_l(e_h) + q$, $\eta'_l(e') = \eta_l(e')$, $\eta'_l(e_l) = \eta_l(e_l) + \kappa - q$, $\eta'_l(1) = \eta_l(1) - \kappa$. Where $\varepsilon < \frac{e_l - e'}{e_h - e_l} \frac{e_h}{e'} \delta$, $\kappa = \frac{q}{e_h} + l(\mu)(\varepsilon + \delta)$, $q = \frac{1 - e_l}{e_h - 1}(\varepsilon + \delta) e_h l(\mu)$, and $\delta > 0$. Choosing ε close to the upper bound and δ close to 0 gives that $(\eta'_h, \eta'_l) \in K_0^E$.

$$\mu \int_E d\eta'_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta'_l(e) - \mu \int_E d\eta_h(e) - \int_E \left(\frac{1}{e} - \mu\right) d\eta_l(e)$$
$$= (1 - \mu)\kappa > 0$$

Case II: $\eta_l(e_l) < e_l l(\mu) \eta_h(e_l)$. Construct (η'_h, η'_l) such that $\eta'_\theta(e) = \eta_\theta(e)$ for all $e \notin \{e_h, e_l, e', 1, 0\}$. Let $\eta'_h(e_h) = \eta_h(e_h) + \varepsilon$, $\eta'_h(e_l) = \eta_h(e_l) - \delta - \varepsilon$, $\eta'_h(e') = \eta_h(e') + \delta$, $\eta'_l(e_h) = \eta_l(e_h) + \kappa - \delta e' l(\mu)$, $\eta'_l(e') = \eta_l(e') + \delta e' l(\mu)$, $\eta'_l(e_l) = \eta_l(e_l)$, $\eta'_l(1) = \eta_l(1) - \kappa$. Where $\varepsilon > \max\{\frac{e_l - e'}{e_h - e_l} \frac{e_h}{e'} \delta, \frac{1 - e_l}{e_h} \delta\}$, $\kappa = \delta l(\mu) \frac{e_h - e_l}{e_h - 1}$, and $\delta > 0$. Choosing ε close to the upper bound and δ close to 0 gives that $(\eta'_h, \eta'_l) \in K_0^E$.

$$\mu \int_E d\eta'_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta'_l(e) - \mu \int_E d\eta_h(e) - \int_E \left(\frac{1}{e} - \mu\right) d\eta_l(e)$$
$$= (1 - \mu)\kappa > 0$$

□

Now we are ready to prove the main result needed for the proof of Theorem 1:

Proposition 2. $\arg\max_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\} \subset \boldsymbol{cvx}(\mathcal{T}(\mathcal{K}_E))$.

Proof. As mentioned before, I prove this result in two cases

Case I: $(\eta_h, \eta_l) \in argmax_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$ such that $\operatorname{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$ and $\frac{\eta_l(e)}{\eta_h(e)} = el(\mu)$ for all $e \in \operatorname{supp}(\eta_h) \cap E \cap [1, \infty]$.

In this case, by proof of claim 7, it must be that $(\eta_h, \eta_l) \in K_1^E$. Let W_I be the set of (η_h, η_l) that satisfy the conditions of case I. Claims 10 -12 establish this case.

Claim 10. *For some* $(\eta_h, \eta_l) \in \arg\max_{K_1^E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$*. If* $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$*, then* $\frac{\eta_l(e)}{\eta_h(e)} \leq 1$ *for all* $e \in E$*.*

Proof. By claim 4 either $\text{supp}(\eta_l) \cap E \cap [\frac{1}{l(\mu)}, \infty] = \varnothing$ or $|\text{supp}(\eta_l) \cap E \cap [1, \infty]| = 1$.

If $\text{supp}(\eta_l) \cap E \cap [1, \infty] = \{e_h\}$ then by $\int_E \left(1 - \frac{1}{e}\right)(\eta_h(e) - d\eta_l(e)) \geq 0$ we get that $\frac{\eta_l(e_h)}{\eta_h(e_h)} \leq 1$.

If $\text{supp}(\eta_h) \cap E \cap [\frac{1}{l(\mu)}, \infty] = \varnothing$ then the claim follows from noting $\frac{\eta_l(e)}{\eta_h(e)} \leq el(\mu)$ for all $e \in E$. □

Define

$$\widetilde{K}_1^E := \{(\eta_h, \eta_l) \in W_I |\ |supp(\eta_h) \cap E \cap [0,1]| \leq 1 \text{ and } \frac{\eta_h(e)}{\eta_l(e)} = el(\mu) \text{ for all } e \in \text{supp}(\eta_h) \cap E \cap [1, \infty]\}$$

Claim 11. *For any* E *we have* $\arg\max_{W_I}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\} \subset \boldsymbol{cvx}(\widetilde{K}_1^E)$*.*

Proof. Fix $(\eta_h, \eta_l) \in \arg\max_{W_I}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$. Assume that $|\text{supp}(\eta_h) \cap E \cap [0,1]| > 2$. By claim 8 we get that $\eta_l(e) = el(\mu)\eta_h(e)$ for all $e \in \text{supp}(\eta_h) \cap E \cap [0,1]$.

By assumption and claim 5 we know that for any $e > 1 \in E$, $\eta_h(e) > 0 \implies \frac{\eta_l(e)}{\eta_h(e)} = el(\mu)$. Pick some $e_l \in \text{supp}(\eta_h) \cap E \cap [0,1]$ Now construct the following (η'_h, η'_l) and (η''_h, η''_l):

$$\eta'_\theta(e) = \frac{\alpha \eta_\theta(e)}{\alpha \int_{E \cap [1,\infty]} d\eta_h(e) + \eta_h(e_l)} \quad \text{for all } e \in E \cap [1, \infty]$$

$$\eta'_\theta(e_l) = \frac{\eta_\theta(e_l)}{\alpha \int_{E \cap [1,\infty]} d\eta_h(e) + \eta_h(e_l)}$$

$$\eta''_\theta(e) = \frac{(1-\alpha)\eta_\theta(e)}{(1-\alpha) \int_{E \cap [1,\infty]} d\eta_h(e) + \int_{E \cap [0,1] \setminus \{e_l\}} d\eta_h(e)} \quad \text{for all } e \in E \cap [1, \infty]$$

$$\eta''_\theta(e) = \frac{\eta_\theta(e)}{(1-\alpha) \int_{E \cap [1,\infty]} d\eta_h(e) + \int_{E \cap [0,1] \setminus \{e_l\}} d\eta_h(e)} \quad \text{for all } e \in E \cap [0,1] \setminus \{e_l\}$$

Note that $\frac{\int_{E \cap [1,\infty]} d\eta_l(e)}{\int_{E \cap [1,\infty]} d\eta_h(e)} \leq 1$ as $W_I \subset K_1^E$, this implies $\frac{\int_{E \cap [1,\infty]} d\eta'_l(e)}{\int_{E \cap [1,\infty]} d\eta'_h(e)} \leq 1$ and $\frac{\int_{E \cap [1,\infty]} d\eta''_l(e)}{\int_{E \cap [1,\infty]} d\eta''_h(e)} \leq 1$. Thus to show that (η'_h, η'_l), $(\eta''_h, \eta''_l) \in W_I$. We need to verify the following sets of inequalities can be satisfied simultaneously:

$$\frac{\left(\frac{1}{e_l}-1\right)\eta_l(e_l)}{\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)d\eta_l(e)} < \alpha < \frac{\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)d\eta_l(e)+\int_{E\cap[0,1]\setminus\{e_l\}}\left(1-\frac{1}{e}\right)d\eta_l(e)}{\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)d\eta_l(e)}$$

$$\frac{\left(\frac{1}{e_l}-1\right)(\eta_h(e_l)-\eta_l(e_l))}{\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)(d\eta_h(e)-d\eta_l(e))} \leq \alpha$$

$$\alpha \leq \frac{\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)(d\eta_h(e)-d\eta_l(e))+\int_{E\cap[0,1]\setminus\{e_l\}}\left(1-\frac{1}{e}\right)(d\eta_h(e)-d\eta_l(e))}{\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)(d\eta_h(e)-d\eta_l(e))}$$

The first two inequalities are satisfied simultaneously for some α as $(\eta_h, \eta_l) \in K_1^E$. Similarly, the third and fourth inequalities also hold simultaneously. Finally, note that the third inequality implies the first one as

$$\frac{\left(\frac{1}{e_l}-1\right)\eta_l(e_l)}{\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)d\eta_l(e)} < \frac{\left(\frac{1}{e_l}-1\right)(\eta_h(e_l)-\eta_l(e_l))}{\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)(d\eta_h(e)-d\eta_l(e))}$$

iff

$$\left(\frac{1}{e_l}-1\right)\eta_l(e_l)\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)d\eta_h(e) < \left(\frac{1}{e_l}-1\right)\eta_h(e_l)\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)d\eta_l(e)$$

which follows from

$$\begin{aligned}&\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)\left[\frac{\eta_l(e_l)}{\eta_h(e_l)}-\frac{\eta_l(e)}{\eta_h(e)}\right]d\eta_h(e)\\ &= l(\mu)\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)[e_l-e]\,d\eta_h(e) < 0\end{aligned}$$

Also, the fourth inequality implies the second one as

$$\begin{aligned}&\int_{E\cap[0,1]\setminus\{e_l\}}\left(1-\frac{1}{e}\right)d\eta_l(e)\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)d\eta_h(e)\\ &> \int_{E\cap[0,1]\setminus\{e_l\}}\left(1-\frac{1}{e}\right)d\eta_h(e)\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)d\eta_l(e)\end{aligned}$$

This follows as

$$\int_{E\cap[0,1]\setminus\{e_l\}}\left(\frac{1}{e'}-1\right)\left[\int_{E\cap[1,\infty]}\left(1-\frac{1}{e}\right)\left[\frac{\eta_l(e')}{\eta_h(e')}-\frac{\eta_l(e)}{\eta_h(e)}\right]d\eta_h(e)\right]d\eta_h(e')$$

$$= l(\mu) \int_{E \cap [0,1] \setminus \{e_l\}} \left(\frac{1}{e'} - 1 \right) \left[\int_{E \cap [1,\infty]} \left(1 - \frac{1}{e} \right) [e' - e] \, d\eta_h(e) \right] d\eta_h(e') < 0$$

Thus we can always choose $\alpha \in [0,1]$ that satisfies the requirements. Thus we get that $(\eta'_h, \eta'_l) \in \widetilde{K}_1^E$ and $(\eta''_h, \eta''_l) \in W_I$. Recursively applying this argument proves the claim. □

Let $\mathcal{B}(\mathcal{K}_E) := \{(\eta_h, \eta_l) \in \mathcal{K}_E | \, |\operatorname{supp}(\eta_h) \cap E| \leq 2\}$

Claim 12. $\widetilde{K}_1^E \subset \boldsymbol{cvx}(\mathcal{B}(\mathcal{K}_E))$.

Proof. If $\operatorname{supp}(\eta_h) \cap [0,1] = \varnothing$ then $(\eta_h, \eta_l) = \sum_{e \in E} \eta_h(e)(\eta_h^e, \eta_l^e)$. Where $\eta_l^e(e) = \frac{\eta_l(e)}{\eta_h(e)}$, $\eta_l^e(1) = 1 - \frac{\eta_l(e)}{\eta_h(e)}$ and $\eta_h^e(e) = 1$ for all $e \in E$.

If $\operatorname{supp}(\eta_h) \cap [0,1] = \{e_l\}$. The claim follows when $|\operatorname{supp}(\eta_h) \cap [1,\infty]| = 1$. Assume that $|\operatorname{supp}(\eta_h) \cap [1,\infty]| > 1$, also for any $e > 1 \in E$, $\eta_h(e) > 0 \implies \frac{\eta_l(e)}{\eta_h(e)} = el(\mu)$. Pick some $e_h \in \operatorname{supp}(\eta_h) \cap [1,\infty]$. Now construct the following (η'_h, η'_l) and (η''_h, η''_l):

$$\eta'_\theta(e_h) = \frac{d\eta_\theta(e_h)}{\alpha\eta_h(e_l) + \eta_h(e_h)}$$

$$\eta'_\theta(e_l) = \frac{\alpha d\eta_\theta(e_l)}{\alpha\eta_h(e_l) + \eta_h(e_)h)}$$

$$\eta''_\theta(e) = \frac{d\eta_\theta(e)}{(1-\alpha)\eta_h(e_l) + \int_{E \cap [1,\infty] \setminus \{e_h\}} d\eta_h(e)} \quad \text{for all } e \in E \cap [1,\infty] \setminus \{e_h\}$$

$$\eta''_\theta(e_l) = \frac{(1-\alpha)\eta(e_l)}{(1-\alpha)\eta_h(e_l) + \int_{E \cap [1,\infty] \setminus \{e_h\}} d\eta_h(e)}$$

By claim 10 we know that $\frac{\eta_l(e)}{\eta_h(e)} \leq 1$ for all $e \in E$. Thus to show that (η'_h, η'_l), $(\eta''_h, \eta''_l) \in \widetilde{K}_1^E$. We need to verify the following sets of inequalities can be satisfied simultaneously:

$$\frac{\left(1 - \frac{1}{e_h}\right) \eta_l(e_h)}{\left(\frac{1}{e_l} - 1\right) \eta_l(e_l)} > \alpha > \frac{\left(\frac{1}{e_l} - 1\right) \eta_l(e_l) + \int_{E \cap [1,\infty] \setminus \{e_h\}} \left(\frac{1}{e} - 1\right) d\eta_l(e)}{\left(\frac{1}{e_l} - 1\right) \eta_l(e_l)}$$

$$\frac{\left(1 - \frac{1}{e_h}\right) (\eta_h(e_h) - d\eta_l(e_h))}{\left(\frac{1}{e_l} - 1\right) (\eta_h(e_l) - \eta_l(e_l))} \geq \alpha$$

$$\alpha \geq \frac{\int_{E \cap [1,\infty] \setminus \{e_h\}} \left(\frac{1}{e} - 1\right) (d\eta_h(e) - d\eta_l(e)) + \left(\frac{1}{e_l} - 1\right) (\eta_h(e_l) - \eta_l(e_l))}{\left(\frac{1}{e_l} - 1\right) (\eta_h(e_l) - \eta_l(e_l))}$$

Note that as $(\eta_h, \eta_l) \in \widetilde{K}_1^E$, the first and second inequalities hold simultaneously for some α, also the third and fourth inequalities hold simultaneously. The claim follows from noting that the third inequality implies the first one and the fourth inequality implies the second one. Third implies the first as

$$\frac{\eta_l(e_h)}{\eta_h(e_h)} = l(\mu)e_h > l(\mu)e_l = \frac{\eta_l(e_l)}{\eta_h(e_l)}$$

and the fourth inequality implies the second one as

$$\frac{\int_{E\cap[1,\infty]\setminus\{e_h\}} \left(1-\frac{1}{e}\right)(d\eta_l(e))}{\int_{E\cap[1,\infty]\setminus\{e_h\}} \left(1-\frac{1}{e}\right)(d\eta_h(e))} \geq l(\mu) > l(\mu)e_l = \frac{\eta_l(e_l)}{\eta_h(e_l)}$$

Thus we can always choose $\alpha \in [0,1]$ that satisfies the requirements. Thus we get that $(\eta_h', \eta_l') \in \mathcal{B}(\mathcal{K}_E)$ and $(\eta_h'', \eta_l'') \in \widetilde{K}_1^E$. Recursively applying this argument proves the claim. □

The proof of the first case then follows from noting that by definition $\mathcal{B}(\mathcal{K}_E) \subset \mathcal{T}(\mathcal{K}_E)$.

Case II: $(\eta_h, \eta_l) \in argmax_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e}-\mu\right) d\widetilde{\eta}_l(e)\}$ such that $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$ and $\frac{\eta_l(e)}{\eta_h(e)} < el(\mu)$ for some $e \in \text{supp}(\eta_h) \cap E \cap [1,\infty]$.

In this case, by proof of claims 6 and 7, it must be that $\int_E \left(1-\frac{1}{e}\right) d\eta_h(e) = \int_E \left(1-\frac{1}{e}\right) d\eta_l(e)$.

Claim 13. *For any* $(\eta_h, \eta_l) \in \arg\max_{K_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e}-\mu\right) d\widetilde{\eta}_l(e)\}$ *such that* $\text{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$. *If there exists some* $e \in \text{supp}(\eta_h) \cap E \cap [1,\infty]$ *such that* $\frac{\eta_l(e)}{\eta_h(e)} < el(\mu)$ *then* $|\text{supp}(\eta_l) \cap E \cap [1,\infty]| = 1$.

Proof. Assume that there exists $e_4 < 1 \in E$ such that $\eta_l(e_4) > 0$. Assume for contradiction that $|\text{supp}(\eta_l) \cap E \cap [1,\infty]| > 1$, then there exist $e_2, e_3 \in \text{supp}(\eta_l) \cap E \cap [1,\infty]$ where $e_2 > e_3$. By claim 5 we get that $\frac{\eta_l(e_2)}{\eta_h(e_2)} = e_2 l(\mu)$ and $\frac{\eta_l(e_3)}{\eta_h(e_3)} = e_3 l(\mu)$. By assumption and claim 5 we get that there exists $e_1 \in \text{supp}(\eta_h) \cap E \cap [1,\infty]$ such that $e_1 > e_2$ and $\eta_l(e_1) = 0$.

Construct (η_h', η_l') such that $\eta_\theta'(e) = \eta_\theta(e)$ for all $e \notin \{e_1, e_2, e_3, e_4, 1\}$. Let $\eta_h'(e_1) = \eta_h(e_1) - q$, $\eta_h'(e_2) = \eta_h(e_2) - \varepsilon$, $\eta_h'(e_4) = \eta_h(e_4) - \delta$, $\eta_h'(e_3) = \eta_h(e_3) + q + \varepsilon + \delta$, $\eta_l'(e_1) = 0$, $\eta_l'(e_2) = \eta_l(e_2) - \varepsilon e_2 l(\mu)$, $\eta_l'(e_4) = \eta_l(e_4) - \delta e_4 l(\mu)$, $\eta_l'(e_3) = \eta_l(e_3) + (q + \varepsilon + \delta)e_3 l(\mu)$, and $\eta_l'(1) = \eta_l(1) - q l(\mu)$. Where $\varepsilon = \kappa q \frac{\frac{e_1-e_3}{e_1} + \frac{e_3-1}{e_4}}{(e_2-e_3)\left(\frac{1}{e_4}-\frac{1}{e_2}\right)}$, $\delta = q\frac{1}{e_3-e_4}\left[\frac{\kappa\frac{e_1-e_3}{e_1} + \frac{e_3-1}{e_2} + \frac{e_3-1}{e_4}(\kappa-1)}{\frac{1}{e_4}-\frac{1}{e_2}}\right]$.

By choosing some $\kappa > 1$ and $q > 0$ small enough we get that $(\eta'_h, \eta'_l) \in \mathcal{K}_E$. The claim follows by noting that

$$\mu \int_E d\eta'_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\eta'_l(e) - \mu \int_E d\eta_h(e) - \int_E \left(\frac{1}{e} - \mu\right) d\eta_l(e)$$

$$= (1-\mu) q l(\mu) > 0$$

□

Claim 14. *For any* $(\eta_h, \eta_l) \in \arg\max_{\mathcal{K}_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$. *If* $|\operatorname{supp}(\eta_h \cap E \cap [1, \infty]| = 1$, *then* $|\operatorname{supp}(\eta_l) \cap E \cap [0,1]| = 1$.

Proof. Let $(\eta_h, \eta_l) \in \arg\max_{\mathcal{K}_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$, such that $\operatorname{supp}(\eta_h \cap E \cap [1, \infty] = \{e_h\}$ and $\frac{\eta_l(e_h)}{\eta_h(e_h)} < e_h l(\mu)$. Assume for contradiction $|\operatorname{supp}(\eta_l) \cap E \cap [0,1]| > 1$, then by claim 13 we get that $\frac{\eta_l(e)}{\eta_h(e)} = el(\mu)$ for all $e \in \operatorname{supp}(\eta_h) \cap E \cap [0,1]$. By proof of claim 6 and 7 we conclude that $\int_E \left(1 - \frac{1}{e}\right) d\eta_h(e) = \int_E \left(1 - \frac{1}{e}\right) d\eta_l(e)$. Also by corollary 1 we get that $\eta_h(e_h) = 1 - \int_{E\cap[0,1]} d\eta_h(e)$. Combining these we get that

$$\eta_l(e_h) = \eta_h(e_h) - \frac{e_h}{e_h - 1} \int_{E\cap[0,1]} \left(\frac{1}{e} - 1\right) (1 - el(\mu)) d\eta_h(e)$$

The feasibility constraint can then be simplified as

$$1 - \int_{E\cap[0,1]} \left(1 + \frac{e_h(1-e)}{e(e_h - 1)}\right) d\eta_h(e) \geq 0$$

The above defines a half space for the feasible choice of η_h. As the η_l is determined completely by the choice of η_h, the objective function can be expressed as a linear functional of η_h alone. In particular, the extreme points of the feasible region are candidate solutions for the optimal η_h. These extreme points are such that either $\eta_h(e_h) = 1$ or $\eta_h(e) = \frac{e(e_h-1)}{e_h-e}$ for some $e \in E \cap [0,1]$ and zero for all other $e' \in E \cap [0,1]$. In particular notice that at optimum $|\operatorname{supp}(\eta_l) \cap E \cap [0,1]| = 1$.

□

Define $\widetilde{K}_0^E$ as the set of all $(\eta_h, \eta_l) \in \mathcal{K}_E$ such that $|\operatorname{supp}(\eta_h) \cap E \cap [1, \infty]| \leq 2$ and $\exists\, e \in \operatorname{supp}(\eta_h) \cap E \cap [1, \infty]$ such that $\frac{\eta_l(e)}{\eta_h(e)} < el(\mu)$

Claim 15. *For any* $(\eta_h, \eta_l) \in \arg\max_{\mathcal{K}_E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$ *such that* $\operatorname{supp}(\eta_l) \cap E \cap [0,1] \neq \varnothing$*. If there exists some* $e \in \operatorname{supp}(\eta_h) \cap E \cap [1,\infty]$ *such that* $\frac{\eta_l(e)}{\eta_h(e)} < el(\mu)$ *then* $(\eta_h, \eta_l) \in \boldsymbol{cvx}(\widetilde{K}_0^E)$*.*

Proof. Define $E_0 = \{e \in E \cap [1,\infty] \mid e \in \operatorname{supp}(\eta_h) \setminus \operatorname{supp}(\eta_l)\}$. If $|E_0| = 1$, then the claim follows from claim 13. Assume that $|E_0| > 1$. By claim 13 we get that $|\operatorname{supp}(\eta_l) \cap E \cap [1,\infty]| = 1$.

Pick some $e_h \in E_0$. Now construct the following (η_h', η_l') and (η_h'', η_l''):

$$\eta_\theta'(e_h) = \frac{\eta_\theta(e_h)}{\alpha \int_{E\setminus E_0} d\eta_h(e) + \eta_h(e_h)}$$

$$\eta_\theta'(e) = \frac{\alpha\eta_\theta(e)}{\alpha \int_{E\setminus E_0} d\eta_h(e) + \eta_h(e_h)} \text{ for all } e \in E \setminus E_0$$

$$\eta_\theta''(e) = \frac{\eta_\theta(e)}{(1-\alpha) \int_{E\setminus E_0} d\eta_h(e) + \int_{E_0\setminus\{e_h\}} d\eta_h(e)} \text{ for all } e \in E_0 \setminus \{e_h\}$$

$$\eta_\theta''(e) = \frac{(1-\alpha)\eta_\theta(e)}{(1-\alpha) \int_{E\setminus E_0} d\eta_h(e) + \int_{E_0\setminus\{e_h\}} d\eta_h(e)} \text{ for all } e \in E \setminus E_0$$

When $\alpha = \frac{\left(1-\frac{1}{e_h}\right)\eta_h(e_h)}{\int_{E\setminus E_0}\left(\frac{1}{e}-1\right)(d\eta_h(e) - d\eta_l(e))}$, we get that $\int_{E\setminus(E_0\setminus\{e_h\})} \left(1 - \frac{1}{e}\right)(d\eta_h'(e) - d\eta_l'(e)) = \int_{E\setminus e_h} \left(1 - \frac{1}{e}\right)(d\eta_h''(e) - d\eta_l''(e)) = 0$. Moreover, by construction $\int_{E\setminus(E_0\setminus\{e_h\})} \left(1 - \frac{1}{e}\right) d\eta_l'(e), \int_{E\setminus e_h} \left(1 - \frac{1}{e}\right) d\eta_l''(e) \geq 0$. Finally $(\eta_h', \eta_l') \in \mathcal{K}_E$ and $(\eta_h'', \eta_l'') \in \mathcal{K}_E$ follows from noting the following:

$$\int_{E\setminus E_0} d\eta_h(e) - d\eta_l(e) \geq \int_{E\setminus E_0} (1 - el(\mu)) d\eta_h(e)$$
$$> \int_{E\setminus E_0} (1-e) d\eta_h = \int_{E\setminus E_0} \frac{l(\mu)e}{l(\mu)} \left(\frac{1}{e} - 1\right) d\eta_h(e)$$
$$= \frac{1}{l(\mu)} \int_{E\setminus E_0} \left(\frac{1}{e} - 1\right) d\eta_l(e) \geq 0$$
$$\implies \int_E d\eta_l'(e), \int_E \eta_l''(e) < 1$$

□

Define

$$\mathcal{T}(\mathcal{K}_E) := \{(\eta_h, \eta_l) \in \mathcal{K}_E \mid |\operatorname{supp}(\eta_h) \cap E| \leq 3, |\operatorname{supp}(\eta_l) \cap E| \leq 2\}$$

Claim 16. $\arg\max_{\widetilde{K}_0^E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\} \subset \mathcal{T}(\mathcal{K}_E)$.

Proof. Fix some $(\eta_h^*, \eta_l^*) \in \arg\max_{\widetilde{K}_0^E}\{\mu \int_E d\widetilde{\eta}_h(e) + \int_E \left(\frac{1}{e} - \mu\right) d\widetilde{\eta}_l(e)\}$. Assume that $|\operatorname{supp}(\eta_l^*) \cap E \cap [0,1]| \geq 2$ otherwise the claim is trivial. When $|\operatorname{supp}(\eta_h^*) \cap E \cap [1,\infty]| < 2$, the claim follows from claim 14. Assume that $\operatorname{supp}(\eta_h) \cap E \cap [1,\infty] = \{\bar{e}, e_h\}$, where $e_h < \bar{e}$. We get by claim 5, claim 8, and claim 9 that $\eta_l^*(e) = el(\mu)$ for all $e \in E \setminus \{\bar{e}\}$ and $\eta_l^*(\bar{e}) = 0$. Thus we get the following:

$$\eta_h(e_h) \geq \frac{1}{e_h - 1}\int_{E\cap[0,1]} (1-e)d\eta_h(e)$$

$$\eta_h(\bar{e}) = 1 - \int_{E\setminus\bar{e}} d\eta_h(e)$$

and

$$\int_E \left(1 - \frac{1}{e}\right) d\eta_h(e) = \int_E \left(1 - \frac{1}{e}\right) d\eta_l(e)$$

$$\implies \eta_h(e_h) = \frac{1 - \frac{1}{\bar{e}} + \int_{E\cap[0,1]} \left(l(\mu)(1-e) - \left(\frac{1}{e} - \frac{1}{\bar{e}}\right)\right) d\eta_h(e)}{\frac{1}{e_h} - \frac{1}{\bar{e}} + l(\mu)(e_h - 1)}$$

and

$$\implies \eta_h(\bar{e}) = 1 - \frac{1 - \frac{1}{\bar{e}}}{\frac{1}{e_h} - \frac{1}{\bar{e}} + l(\mu)(e_h-1)} - \int_{E\cap[0,1]} \frac{l(\mu)(e_h - e) - \left(\frac{1}{e} - \frac{1}{{}_h e}\right)}{\frac{1}{e_h} - \frac{1}{\bar{e}} + l(\mu)(e_h-1)}$$

In particular, the feasibility requires

$$\frac{1}{e_h} - \frac{1}{\bar{e}} + l(\mu)(e_h - 1) \overset{(1)}{\geq} 1 - \frac{1}{\bar{e}} + \int_{E\cap[0,1]} \left(l(\mu)(1-e) - \left(\frac{1}{e} - \frac{1}{\bar{e}}\right)\right) d\eta_h(e) \overset{(2)}{\geq} 0$$

$$\left(l(\mu) - \frac{1}{e_h}\right)(e_h - 1) \overset{(3)}{\geq} \int_{E\cap[0,1]} \left(l(\mu) - \frac{1}{ee_h}\right)(e_h - e)d\eta_h(e)$$

$$1 \overset{(4)}{\geq} \frac{1}{(\bar{e}-1)(e_h-1)} \int_{E\cap[0,1]} \left(\frac{(1-e)(\bar{e}-e_h)}{e_h} + \frac{(\bar{e}-e)(e_h-1)}{e}\right) d\eta_h(e)$$

Note that (4) $\implies$ (2), also (2) & (3) $\implies$ (1), also note that (4) simplifies to $1 \geq \int_{E\cap[0,1]} \frac{e_h - e}{e_h - 1}\left(\frac{\bar{e}(e_h-1)+e(\bar{e}-e_h)}{e_h e(\bar{e}-1)}\right) d\eta_h(e)$ and that $\frac{e_h-e}{e_h-1}\left(\frac{\bar{e}(e_h-1)+e(\bar{e}-e_h)}{e_h e(\bar{e}-1)}\right) > 1$ for all $e \in E \cap [0,1]$.

If $l(\mu) \geq \frac{1}{e_h}$ then (4) $\implies$ (3). This follows from:

$$\frac{l(\mu)-\frac{1}{ee_h}}{l(\mu)-\frac{1}{e_h}} \leq \frac{\bar{e}(e_h-1)+e(\bar{e}-e_h)}{e_h e(\bar{e}-1)} \text{ for all } e \in E \cap [0,1]$$

Denote the first coordinate of $\widetilde{K}_0^E$ by J_0^E. Thus the optimization problem can be relaxed as

$$\max_{\eta_h \in J_0^E} T(\eta_h)$$

$$\text{such that } G(\eta_h) \geq 0$$

Where $G(\eta_h) \geq 0$ defines a closed convex half space as $G(\eta_h) = 1 - \int_{E\cap[0,1]} \frac{e_h-e}{e_h-1}\left(\frac{\bar{e}(e_h-1)+e(\bar{e}-e_h)}{e_h e(\bar{e}-1)}\right) d\eta_h(e)$. The feasible region $G(\eta_h) \geq 0$ is closed and convex with extreme points given by η_h such that $\int_{E\cap[0,1]} d\eta_h(e) = 0$ or $\eta_h(e) = \frac{e_h-1}{e_h-e}\left(\frac{e_h e(\bar{e}-1)}{\bar{e}(e_h-1)+e(\bar{e}-e_h)}\right)$ for some $e \in E \cap [0,1]$ and $\eta_h(e') = 0$ for all $e' \in E \cap [0,1] \setminus \{e\}$.

If $l(\mu) < \frac{1}{e_h}$ then (3) is given by

$$1 \leq \int_{E\cap[0,1]} \frac{l(\mu)-\frac{1}{ee_h}}{l(\mu)-\frac{1}{e_h}} \frac{e_h-e}{e_h-1} d\eta_h(e)$$

Thus the optimization problem is given by

$$\max_{\eta_h \in J_0^E} T(\eta_h)$$

$$\text{such that } G(\eta_h) \geq 0$$

$$H(\eta_h) \geq 0$$

Where $H(\eta_h) = \int_{E\cap[0,1]} \frac{l(\mu)-\frac{1}{ee_h}}{l(\mu)-\frac{1}{e_h}} \frac{e_h-e}{e_h-1} d\eta_h(e) - 1$. The halfspaces defined by (3) and (4) have none empty intersection as

$$\frac{l(\mu)-\frac{1}{ee_h}}{l(\mu)-\frac{1}{e_h}} > \frac{\bar{e}(e_h-1)+e(\bar{e}-e_h)}{e_h e(\bar{e}-1)} \text{ for all } e \in E \cap [0,1]$$

Thus, the feasible region $\{\eta_h | G(\eta_h) \geq 0 \geq -H(\eta_h)\}$ is closed, nonempty and convex with extreme points given by η_h such that $\eta_h(e) = \frac{e_h-1}{e_h-e}\left(\frac{e_h e(\bar{e}-1)}{\bar{e}(e_h-1)+e(\bar{e}-e_h)}\right)$ or $\eta_h(e) = \frac{l(\mu)-\frac{1}{e_h}}{l(\mu)-\frac{1}{ee_h}} \frac{e_h-1}{e_h-e}$ for some $e \in E \cap [0,1]$ and $\eta_h(e') = 0$ for $e' \in E \cap [0,1] \setminus \{e\}$. Here the first type of extreme points are on the hyperplane $G(\eta_h) = 0$ and the second type lies on the hyperplane $H(\eta_h) = 0$.

The objective

$$T(\eta_h) = \mu \int_E d\eta_h(e) + \int_{E\setminus\{\bar{e}\}} \left(\frac{1}{e} - \mu\right) el(\mu) d\eta_h(e)$$

$$= \text{constant} + \int_{E\cap[0,1]} l(\mu) \left((1-\mu_h) \frac{l(\mu)(1-e) - \frac{1}{e} + \frac{1}{\bar{e}}}{\frac{1}{e_h} - \frac{1}{\bar{e}} + l(\mu)(e_h - 1)} + (1 - e\mu) \right) d\eta_h(e)$$

This is linear in η_h thus the optimum is achieved at an extreme point, thus we prove the claim as for any extreme point it holds that $|\operatorname{supp}(\eta_h) \cap E \cap [0,1]| \leq 1$.

□

Claims 10 -12 establish case I, and claims 13 - 16 establish case II. Thus, proposition 2 follows by combining these with claims 2 and 3. □

Theorem 1 follows from Proposition 2 by constructing the menu such that $\sigma_h(.|h) = \eta_h(.)$ and $\sigma_h(.|h) = \eta_l(.)$ with payments according to Proposition 1.

6.8 Corollary 3

I prove a stronger statement than the statement of Corollary 3 presented in the text. In particular, I prove a characterization of equilibrium with binary acceptance set. The first two experiment structures are enumerated in the case when soft information overrides hard information.

Lemma 5. *If $E = \{e_h, e_l\}$, where $e_l < 1$ and $e_h > 1$. Then $(\sigma_h, \sigma_l) \in \mathcal{K}_E$ if and only if it has the following form:*
If $e_h \leq \min\{\frac{1}{l(\mu)}, \frac{1}{\mu}\}$ and $\frac{1}{\mu+(1-\mu)l(\mu)e_l} \geq e_h$

$$\sigma_h$$

$$\begin{pmatrix} \frac{e_h(1-e_l)}{e_h-e_l}\frac{1-l(\mu)e_l}{1-l(\mu)e_he_l} & \frac{e_l(e_h-1)}{e_h-e_l}\frac{1-l(\mu)e_h}{1-l(\mu)e_he_l} & 0 \\ \frac{1-e_l}{e_h-e_l}\frac{1-l(\mu)e_l}{1-l(\mu)e_he_l} & \frac{e_h-1}{e_h-e_l}\frac{1-l(\mu)e_h}{1-l(\mu)e_he_l} & \frac{l(\mu)(e_h+e_l-e_he_l-1)}{1-l(\mu)e_he_l} \end{pmatrix}$$

$$\sigma_l$$

$$\begin{pmatrix} e_hl(\mu)\frac{e_h(1-e_l)}{e_h-e_l}\frac{1-l(\mu)e_l}{1-l(\mu)e_he_l} & e_ll(\mu)\frac{e_l(e_h-1)}{e_h-e_l}\frac{1-l(\mu)e_h}{1-l(\mu)e_he_l} & 1-l(\mu)\left[e_l - e_h(1-e_l)\frac{1-l(\mu)e_l}{1-l(\mu)e_he_l}\right] & 0 \\ e_hl(\mu)\frac{1-e_l}{e_h-e_l}\frac{1-l(\mu)e_l}{1-l(\mu)e_he_l} & e_ll(\mu)\frac{e_h-1}{e_h-e_l}\frac{1-l(\mu)e_h}{1-l(\mu)e_he_l} & 1-l(\mu)\left[e_l - e_h(1-e_l)\frac{1-l(\mu)e_l}{1-l(\mu)e_he_l}\right] & l(\mu)\frac{(e_h-1)(1-e_l)}{1-l(\mu)e_he_l} \end{pmatrix}$$

If $e_h > \frac{1}{\mu}$ *or if* $\frac{1}{l(\mu)} < e_h \le \frac{1}{\mu}$ *and* $\frac{1}{e_h} \le \mu + (1-\mu)e_l l(\mu)$ *or if* $e_h \le \min\{\frac{1}{l(\mu)}, \frac{1}{\mu}\}$ *and* $\frac{1}{\mu+(1-\mu)l(\mu)e_l} < e_h$

$$\overset{\sigma_h}{\begin{pmatrix} \frac{e_h(1-e_l)}{e_h-e_l} & \frac{e_l(e_h-1)}{e_h-e_l} \\ \frac{1-e_l}{e_h-e_l} & \frac{e_h-1}{e_h-e_l} \end{pmatrix}} \qquad \overset{\sigma_l}{\begin{pmatrix} e_l l(\mu)\frac{e_h(1-e_l)}{e_h-e_l} & e_l l(\mu)\frac{e_l(e_h-1)}{e_h-e_l} & 1-l(\mu)e_l \\ e_l l(\mu)\frac{1-e_l}{e_h-e_l} & e_l l(\mu)\frac{e_h-1}{e_h-e_l} & 1-l(\mu)e_l \end{pmatrix}}$$

and, if $\frac{1}{l(\mu)} < e_h \le \frac{1}{\mu}$ *and* $\frac{1}{e_h} > \mu + (1-\mu)e_l l(\mu)$

$$\overset{\sigma_h=\sigma_l}{\begin{pmatrix} 1 & 0 \\ \frac{1}{e_h} & 1-\frac{1}{e_h} \end{pmatrix}}$$

Proof. A binary menu wrt $E = \{e_h(>1), e_l(<1)\}$ is of the form the following form for some $x \in (0,1]$.

$$\sigma_h \begin{pmatrix} x & 1-x & 0 \\ \frac{x}{e_h} & \frac{1-x}{e_l} & 1-\frac{x}{e_h}-\frac{1-x}{e_l} \end{pmatrix}$$

$$\sigma_l \begin{pmatrix} \alpha x & \beta(1-x) & 1-\alpha x-\beta(1-x) & 0 \\ \alpha\frac{x}{e_h} & \beta\frac{1-x}{e_l} & 1-\alpha x-\beta(1-x) & \alpha\left(1-\frac{1}{e_h}\right)x+\beta\left(1-\frac{1}{e_l}\right)(1-x) \end{pmatrix}$$

Thus, the optimization problem in Proposition 1 is simplified to

$$\max\ \mu + \left(\frac{1}{e_h}-\mu\right)\alpha x + \left(\frac{1}{e_l}-\mu\right)\beta(1-x)$$

subject to
1) $x \in [0,1]$, 2) $\frac{1}{e_h}x + \frac{1}{e_l}(1-x) \le 1$, 3) $\alpha x\left(1-\frac{1}{e_h}\right) + \beta(1-x)\left(1-\frac{1}{e_l}\right) \ge 0$, 4) $\alpha x + \beta(1-x) \le 1$, 5) $\alpha x\left(1-\frac{1}{e_h}\right) + \beta(1-x)\left(1-\frac{1}{e_l}\right) \ge 0$, 6) $\alpha \le \min\{e_h l(\mu), 1\}$ and 7) $\beta \le l(\mu)e_l$

Note that 2) $\frac{1}{e_h}x + \frac{1}{e_l}(1-x) \le 1$ implies $x \ge \frac{e_h(1-e_l)}{e_h-e_l}$. 3) $\alpha x\left(1-\frac{1}{e_h}\right) + \beta(1-x)\left(1-\frac{1}{e_l}\right) \ge 0$ implies $\beta \le \alpha\frac{e_l(e_h-1)}{e_h(1-e_l)}\frac{x}{1-x}$. 4) $\alpha x + \beta(1-x) \le 1$, implies that $\alpha \le 1 - \frac{e_h(1-e_l)}{e_l(e_h-1)}\frac{1-x}{x}(1-\beta)$.

If $\frac{1}{e_h} < \mu$ then the revenue is decreasing in α and x. Setting $x = \frac{e_h(1-e_l)}{e_h-e_l}$, we get that $\alpha = \beta$ by 3) and 4). Thus the revenue is $\mu + \left(\frac{1}{e_l} - \mu - \frac{e_h(1-e_l)}{e_h-e_l}\left(\frac{1}{e_l}-\frac{1}{e_h}\right)\right)\beta = \mu + \left(\frac{1}{e_l} - \mu - \frac{1-e_l}{e_l}\right)\beta$. Thus revenue is increasing in β. Hence the optimal experiment is such that $\sigma_h(e_h) = \frac{e_h(1-e_l)}{e_h-e_l}$ and $\alpha = \beta = e_l l(\mu)$.

If $l(\mu) > \frac{1}{e_h} > \mu$ the revenue is increasing in α. Thsu or fixed x, β we have $\alpha = \min\{e_h l(\mu), 1 - \frac{e_h(1-e_l)}{e_l(e_h-1)} \frac{1-x}{x}(1-\beta)\}$. As $e_h l(\mu)) > 1$ we get that $\alpha = 1 - \frac{e_h(1-e_l)}{e_l(e_h-1)} \frac{1-x}{x}(1-\beta)$ Then the revenue is given by:

$$\mu + (1 - \frac{e_h(1-e_l)}{e_l(e_h-1)} \frac{1-x}{x}(1-\beta))x \left(\frac{1}{e_h} - \mu\right) + \left(\frac{1}{e_l} - \mu\right) \beta(1-x)$$

$$= \mu - \frac{e_h(1-e_l)}{e_l(e_h-1)} \left(\frac{1}{e_h} - \mu\right) + \beta \left(\frac{1}{e_l} - \mu\right) + x \left[\left(\frac{1}{e_h} - \mu\right)(1 + \frac{e_h(1-e_l)}{e_l(e_h-1)}(1-\beta)) - \beta \left(\frac{1}{e_l} - \mu\right)\right]$$

$$= \mu - \frac{e_h(1-e_l)}{e_l(e_h-1)} \left(\frac{1}{e_h} - \mu\right) + \beta \left(\frac{1}{e_l} - \mu\right)$$
$$+ x \left[\left(\frac{1}{e_h} - \mu\right)\left(1 + \frac{e_h(1-e_l)}{e_l(e_h-1)}\right) - \beta \left(\frac{1}{e_l} - \mu + \frac{e_h(1-e_l)}{e_l(e_h-1)} \left(\frac{1}{e_h} - \mu\right)\right)\right]$$

$$= \mu - \frac{e_h(1-e_l)}{e_l(e_h-1)} \left(\frac{1}{e_h} - \mu\right) + \beta \left(\frac{1}{e_l} - \mu\right) + x \frac{e_h - e_l}{e_l(e_h-1)} \left[\frac{1}{e_h} - \mu - \beta(1-\mu)\right]$$

Whenever $\frac{1}{e_h} - \mu - \beta(1-\mu) > 0$ then the revenue is increasing in x, thus the optimal experiment is such that $x = 1$ and $\alpha = 1$. Whenever $\frac{1}{e_h} - \mu - \beta(1-\mu) < 0$ then revenue is decreasing in x for fixed β. Thus $x = \frac{e_h(1-e_l)}{e_h - e_l}$, this gives us that the optimal experiment is such that $\sigma_h(e_h) = \frac{e_h(1-e_l)}{e_h - e_l}$ and $\alpha = \beta = e_l l(\mu)$.

If $e_h \leq \frac{1}{\mu}, \frac{1}{l(\mu)}$, the revenue is increasing in β and α. For fixed x, α, we get that $\beta = \min\{e_l(l(\mu), \alpha \frac{e_l(e_h-1)}{e_h(1-e_l)} \frac{x}{1-x}\}$. If $\beta = \alpha \frac{e_l(e_h-1)}{e_h(1-e_l)} \frac{x}{1-x}$, then the revenue is given by

$$\mu + x\alpha \left[\frac{1}{e_h} - \mu + \left(\frac{1}{e_l} - \mu\right) \frac{e_l(e_h-1)}{e_h(1-e_l)}\right]$$
$$= \mu + \alpha x \frac{e_h - e_l}{e_h(1-e_l)}(1-\mu)$$

The expression is increasing in x. As $\alpha \frac{e_l(e_h-1)}{e_h(1-e_l)} \frac{x}{1-x} < e_l l(\mu)$ we get that

$$x \leq \frac{l(\mu) e_h(1-e_l)}{\alpha(e_h-1) + l(\mu) e_h(1-e_l))}$$

As the revenue is increasing in both α and x, thus for fixed α set $x = \frac{l(\mu) e_h(1-e_l)}{\alpha(e_h-1) + l(\mu) e_h(1-e_l)}$. Then the revenue is given by

$$\mu + (1-\mu) l(\mu) \alpha \frac{e_h - e_l}{\alpha(e_h-1) + l(\mu) e_h(1-e_l)}$$

This expression is increasing in α. As $x \geq \frac{e_h(1-e_l)}{e_h-e_l}$ we get that

$$\frac{l(\mu)e_h(1-e_l)}{\alpha(e_h-1)+l(\mu)e_h(1-e_l)} \geq \frac{e_h(1-e_l)}{e_h-e_l}$$
$$\implies l(\mu)(e_h-e_l) \geq \alpha(e_h-1)+l(\mu)e_h(1-e_l)$$
$$\implies \alpha \leq l(\mu)e_l$$

Thus the optimal experiment has $x = \frac{e_h(1-e_l)}{e_h-e_l}$ and $\alpha = \beta = e_l l(\mu)$.

If $\beta = e_l l(\mu)$ then the revenue is increasing in α and x. In particular for fixed x we get that $x\alpha \geq (1-x)\frac{e_h(1-e_l)}{e_l(e_h-1)}$ and $\alpha = \min\{e_h l(\mu), 1 - \frac{e_h(1-e_l)}{e_l(e_h-1)}\frac{1-x}{x}(1-e_l l(\mu))\}$. For $x \geq \frac{e_h(1-e_l)}{e_h-e_l}\frac{1-l(\mu)e_l}{1-l(\mu))e_l e_h}$, we get that $\alpha = e_h l(\mu)$, in which case the revenue is

$$\mu + e_l l(\mu)\left(\frac{1}{e_l}-\mu\right) + l(\mu)x\mu(e_l-e_h)$$

this is decreasing in x thus it's optimal to set $x = \frac{e_h(1-e_l)}{e_h-e_l}\frac{1-l(\mu)e_l}{1-l(\mu))e_l e_h}$, $\alpha = e_h l(\mu)$ and $\beta = e_l l(\mu)$.

If $x < \frac{e_h(1-e_l)}{e_h-e_l}\frac{1-l(\mu)e_l}{1-l(\mu))e_l e_h}$ we get $\alpha = 1 - \frac{e_h(1-e_l)}{e_l(e_h-1)}\frac{1-x}{x}(1-e_l l(\mu))$, then the revenue is given by

$$= \mu - \frac{e_h(1-e_l)}{e_l(e_h-1)}\left(\frac{1}{e_h}-\mu\right) + \beta\left(\frac{1}{e_l}-\mu\right) + x\frac{e_h-e_l}{e_l(e_h-1)}\left[\frac{1}{e_h}-\mu-\beta(1-\mu)\right]$$

Whenever $\frac{1}{e_h}-\mu-\beta(1-\mu) > 0$ then the revenue is increasing in x, thus the optimal experiment is such that $x = \frac{e_h(1-e_l)}{e_h-e_l}\frac{1-l(\mu)e_l}{1-l(\mu))e_l e_h}$, $\alpha = e_h l(\mu)$ and $\beta = e_l l(\mu)$. Whenever $\frac{1}{e_h}-\mu-\beta(1-\mu) < 0$ then revenue is decreasing in x. Thus $x = \frac{e_h(1-e_l)}{e_h-e_l}$, this gives us that the optimal experiment is such that $\sigma_h(e_h) = \frac{e_h(1-e_l)}{e_h-e_l}$ and $\alpha = \beta = e_l l(\mu)$.

Finally to conclude note that for $e_h \leq \frac{1}{l(\mu)}, \frac{1}{\mu}$ the revenue for $x = \frac{e_h(1-e_l)}{e_h-e_l}\frac{1-l(\mu)e_l}{1-l(\mu))e_l e_h}$, $\alpha = e_h l(\mu)$ and $\beta = e_l l(\mu)$ is given by

$$\mu + (1-\mu)l(\mu) - \mu l(\mu)\frac{(e_h-1)(1-e_l)}{1-l(\mu)e_h e_l}$$

revenue when $x = \frac{e_h(1-e_l)}{e_h-e_l}$ and $\alpha = \beta = e_l l(\mu)$ is given by

$$\mu + (1-\mu)l(\mu)e_l$$

Note that

$$\mu + (1-\mu)l(\mu) - \mu l(\mu)\frac{(e_h-1)(1-e_l)}{1-l(\mu)e_h e_l} \geq \mu + (1-\mu)l(\mu)e_l$$

$$\iff (1-\mu) \geq \mu \frac{e_h - 1}{1 - l(\mu) e_h e_l}$$
$$\iff 1 \geq e_h(\mu + (1-\mu) l(\mu) e_l)$$

□

6.9 Corollary 4: Existence of Partial Pooling

By Theorem 1, Corollary 2, and Lemma 6, we get that if m is a menu with partial pooling, then it must be that $m \in \mathbf{cvx}(\mathcal{T}(E)) \setminus \mathbf{cvx}(\mathcal{B}(E))$. Thus, I will focus on menus in the set $\mathcal{T}(E) \setminus \mathcal{B}(E)$. by proposition 3 we get that i f $m \in \mathcal{M}_E^r \cap [\mathcal{T}(E) \setminus \mathcal{B}(E)]$ then $\text{supp}(\sigma_m(.|h)) \cap E = \{\bar{e}, e_h, e_l\}$ and $\text{supp}(\sigma_m(.|l)) \cap E = \{e_h, e_l\}$ where $\bar{e} > e_h > 1 > e_l$. Also, $\frac{\sigma_m(e_h|l)}{\sigma_m(e_h|h)} = e_h l(\mu)$. To establish existence I will assume $\min\{l(\mu), \mu + (1-\mu) l(\mu) \frac{e_h e(\bar{e}-1)}{\bar{e}(e_h-1)+e(\bar{e}-e_h)}\} \geq \frac{1}{e_h}$. Thus by proof of claim 16, we get the optimal menu has the following form for some $x \in (0, 1]$.

$$\sigma_h \begin{pmatrix} 1-x-y & y & x \\ \frac{1-x-y}{\bar{e}} & \frac{y}{e_h} & \frac{x}{e_l} \end{pmatrix}$$

$$\sigma_l \begin{pmatrix} 0 & y e_h l(\mu) & \beta x & 1 - y e_h l(\mu) - \beta x \\ 0 & y l(\mu) & \frac{\beta x}{e_l} & 1 - y e_h l(\mu) - \beta x \end{pmatrix}$$

In particular we get that $\beta = \frac{e_h - 1}{1 - e_l} e_l \frac{y}{x} l(\mu)$ and $y = \frac{1 - \frac{1}{\bar{e}} - \left(\frac{1}{e_l} - \frac{1}{\bar{e}}\right) x}{\frac{1}{e_h} - \frac{1}{\bar{e}}}$. The revenue is given by

$$\mu + (1-\mu) l(\mu) y \frac{e_h - e_l}{1 - e_l}$$

Note that we need $x + y \leq 1$, $e_h l(\mu) y + \beta x \leq 1$ and $\beta \leq e_l l(\mu)$. Form $e_h l(\mu) y + \beta x \leq 1$ we get that

$$l(\mu) y \left(e_h + \frac{e_h - 1}{1 - e_l} e_l \right) \leq 1$$
$$\implies y \leq \frac{1}{l(\mu)} \frac{1 - e_l}{e_h - e_l}$$
$$\implies x \geq \frac{1 - \frac{1}{\bar{e}} - \frac{1}{l(\mu)} \frac{1 - e_l}{e_h - e_l} \left(\frac{1}{e_h} - \frac{1}{\bar{e}} \right)}{\frac{1}{e_l} - \frac{1}{\bar{e}}}$$

From $\beta \leq l(\mu) e_l$ we get that

$$1 - \frac{1}{\bar{e}} - \left(\frac{1}{e_l} - \frac{1}{\bar{e}} \right) x \leq x \left(\frac{1}{e_h} - \frac{1}{\bar{e}} \right) \frac{1 - e_l}{e_h - 1}$$

$$\implies x \geq \frac{1-\frac{1}{\bar{e}}}{\left(\frac{1}{e_l}-\frac{1}{\bar{e}}\right)+\left(\frac{1}{e_h}-\frac{1}{\bar{e}}\right)\frac{1-e_l}{e_h-1}}$$

From $x+y\leq 1$ we get that

$$\frac{1-\frac{1}{\bar{e}}-\left(\frac{1}{e_l}-\frac{1}{e_h}\right)x}{\frac{1}{e_h}-\frac{1}{\bar{e}}}\leq 1$$

$$\implies x \geq \frac{1-\frac{1}{e_h}}{\frac{1}{e_l}-\frac{1}{e_h}}=\frac{e_l(e_h-1)}{e_h-e_l}$$

The revenue is decreasing in x and the following inequalities hold $\frac{1-\frac{1}{\bar{e}}}{\left(\frac{1}{e_l}-\frac{1}{\bar{e}}\right)+\left(\frac{1}{e_h}-\frac{1}{\bar{e}}\right)\frac{1-e_l}{e_h-1}} \leq \frac{1-\frac{1}{\bar{e}}}{\left(\frac{1}{e_l}-\frac{1}{\bar{e}}\right)+\left(\frac{1}{e_h}-\frac{1}{\bar{e}}\right)\frac{1-e_l}{e_h-1}} \leq 1$ and $\frac{1-\frac{1}{\bar{e}}-\frac{1}{l(\mu)}\frac{1-e_l}{e_h-e_l}\left(\frac{1}{e_h}-\frac{1}{\bar{e}}\right)}{\frac{1}{e_l}-\frac{1}{\bar{e}}} \leq 1$. Note that $\frac{1-\frac{1}{\bar{e}}}{\left(\frac{1}{e_l}-\frac{1}{\bar{e}}\right)+\left(\frac{1}{e_h}-\frac{1}{\bar{e}}\right)\frac{1-e_l}{e_h-1}} \geq \frac{1-\frac{1}{\bar{e}}-\frac{1}{l(\mu)}\frac{1-e_l}{e_h-e_l}\left(\frac{1}{e_h}-\frac{1}{\bar{e}}\right)}{\frac{1}{e_l}-\frac{1}{\bar{e}}}$ whenever $l(\mu) \leq \frac{e_l(\bar{e}-e_h)+\bar{e}(e_h-1)}{(\bar{e}-1)e_l e_h}$. Finally by noting that $\frac{e_l(\bar{e}-e_h)+\bar{e}(e_h-1)}{(\bar{e}-1)e_l e_h} \geq 1$ we get that $x=\frac{1-\frac{1}{\bar{e}}}{\left(\frac{1}{e_l}-\frac{1}{\bar{e}}\right)+\left(\frac{1}{e_h}-\frac{1}{\bar{e}}\right)\frac{1-e_l}{e_h-1}}$, $y=\frac{1-\frac{1}{\bar{e}}}{\left(\frac{1}{e_l}-\frac{1}{\bar{e}}\right)\frac{e_h-1}{1-e_l}+\left(\frac{1}{e_h}-\frac{1}{\bar{e}}\right)}$ and $\beta = l(\mu)e_l$.