Supplementary Material: Incentive-Compatible Information Design*

Hershdeep Chopra †　　Jeffrey C. Ely ‡

February 17, 2026

Abstract

This supplement contains ommitted proofs and auxiliary results.

1 Proof of Theorem 1

Proof of Theorem 1. Assume for contradiction that there exists an individually rational, obedient direct mechanism (ρ, ϕ) such that there exists positive measure (with respect to F) subsets of $[0,1]$, S_1 and S_2 , where $S_2 >_{SSO} S_1$ in the strong set order [1] and $\operatorname{ess\,inf}_{S_1} K(\mu) > \operatorname{ess\,sup}_{S_2} K(\mu)$. Consequently we can choose positive measure (with respect to F) subsets $S_1' \subseteq S_1$ and $S_2' \subseteq S_2$ such that one of the following conditions holds

$$\inf_{\mu \in S_1'} \mathbf{E}_A \left[\pi_S(a,1) \rho_1(\mu)(a) \right] > \sup_{\mu \in S_2'} \mathbf{E}_A \left[\pi_S(a,1) \rho_1(\mu)(a) \right] \qquad (\ \theta = 1\ )$$

and

$$\int_{S_1'} \mu \, dF(\mu) = \int_{S_2'} \mu \, dF(\mu)$$

---

*The recent version of the paper can be found here.

†Department of Economics, Northwestern University. hershdeepchopra2026@u.northwestern.edu.

‡Department of Economics, Northwestern University. jeff@jeffely.com. Support from the National Science Foundation is gratefully, albeit nervously, acknowledged.

[1] Let Y and Y' be subsets of $\mathbb{R}$. Set Y' dominates Y in the strong set order ($Y' >_{SSO} Y$) if for any x' in Y' and x in Y, we have $\max\{x', x\}$ in Y' and $\min\{x', x\}$ in Y.

or

$$\sup_{\mu \in S_1'} \mathbf{E}_A \left[\pi_S(a, 0) \rho_0(\mu)(a) \right] < \inf_{\mu \in S_2'} \mathbf{E}_A \left[\pi_S(a, 0) \rho_0(\mu)(a) \right] \tag{θ= 0}$$

and

$$\int_{S_1'} 1 - \mu \, dF(\mu) = \int_{S_2'} 1 - \mu \, dF(\mu)$$

Define the following push forward measure (interpreting F both as the CDF and the corresponding measure)

$$\gamma_0 = (1 - \mu)_{\#} \, F \quad \text{and} \quad \gamma_1 = \mu_{\#} \, F$$

We can then define the conditional distributions for each set S_1' and S_2'

$$\gamma_\theta^i = \frac{\gamma_\theta(S_i' \cap \cdot)}{\gamma_\theta(S_i')}$$

Note that as F is continuous, the measure γ_θ^i is atomless. Thus by Theorem 2.5 in **?** we get that there exists unique monotone mappings, T_θ^i, indexed by θ and $i \in \{1, 2\}$ such that for all θ and $i \neq j$ we have the following

$$(T_\theta^j)_{\#} \, \gamma_\theta^i = \gamma_\theta^j$$

Depending on one of the exhaustive cases Equation θ= 1 or Equation θ= 0 , we can define a new test ρ' for some given $\varepsilon > 0$ by perturbing ρ_θ on sets S_1' and S_2' as following

$$\rho_\theta'(\mu) = \begin{cases} (1 - \varepsilon)\rho_\theta(\mu) + \varepsilon \rho_\theta(\, T_\theta^2(\mu) \,) & \mu \in S_1' \\ (1 - \varepsilon)\rho_\theta(\mu) + \varepsilon \rho_\theta(\, T_\theta^1(\mu) \,) & \mu \in S_2' \\ \rho_\theta(\mu) & \text{otherwise} \end{cases}$$

By the equality in the cases Equation θ= 1 or Equation θ= 0 and the properties of the mapping T_θ^i, we get that the above perturbed test ρ' is obedient. In the background, we can adjust the fee ϕ to maintain the sender's indirect utility by transferring payments between types whose gross surplus is affected by the perturbed test, this is feasible because of the equality constraints in cases Equation θ= 1 or Equation θ= 0 . Hence, the perturbed test also corresponds to an individually rational mechanism. We can establish the contradiction by showing that the perturbed allocation strictly improves the expression in Equation 3 Consider the change in Equation 3 from the perturbation in either of the cases Equation $\theta = 1$ or Equation $\theta = 0$

$$\varepsilon \int_{S_1'} \frac{1-F(\mu)}{f(\mu)} \left(\int_A \pi_S(a,\theta) \, d\left[\rho_\theta(T_\theta^2(\mu))(a) - \rho_\theta(\mu)(a)\right] \right) dF(\mu) + \varepsilon \int_{S_2'} \frac{1-F(\mu)}{f(\mu)} \left(\int_A \pi_S(a,\theta) \, d\left[\rho_\theta(T_\theta^1(\mu))(a) - \rho_\theta(\mu)(a)\right] \right) dF(\mu)$$

We proceed with the case Equation $\theta = 0$, the proof of the other case is analogous. Dividing and multiplying the integrand of the above equation by $1-\mu$ yields the following

$$\varepsilon \int_{S_1'} \frac{1-F(\mu)}{(1-\mu)f(\mu)} \left(\int_A \pi_S(a,0) \, d\left[\rho_0(T_0^2(\mu))(a) - \rho_0(\mu)(a)\right] \right) d\gamma_0(\mu) + \varepsilon \int_{S_2'} \frac{1-F(\mu)}{(1-\mu)f(\mu)} \left(\int_A \pi_S(a,0) \, d\left[\rho_0(T_0^1(\mu))(a) - \rho_0(\mu)(a)\right] \right) d\gamma_0(\mu)$$

Dividing the above expression by $\gamma_0(S_1') \times \gamma_0(S_2')$

$$= \varepsilon \frac{1}{\gamma_0(S_2')} \int_{S_1'} \frac{1-F(\mu)}{(1-\mu)f(\mu)} \left(\int_A \pi_S(a,\theta) \, d\left[\rho_0(T_0^2(\mu))(a) - \rho_0(\mu)(a)\right] \right) d\gamma_0^1(\mu) + \varepsilon \frac{1}{\gamma_0(S_1')} \int_{S_2'} \frac{1-F(\mu)}{(1-\mu)f(\mu)} \left(\int_A \pi_S(a,\theta) \, d\left[\rho_0(T_0^1(\mu))(a) - \rho_0(\mu)(a)\right] \right) d\gamma_0^2(\mu)$$

As $\gamma_0(S_1') = \gamma_0(S_2') > 0$ and $\varepsilon > 0$ the above expression is positive if and only if the following holds

$$= \int_{S_1'} \frac{1-F(\mu)}{(1-\mu)f(\mu)} \left(\int_A \pi_S(a,\theta) \, d\left[\rho_0(T_0^2(\mu))(a) - \rho_0(\mu)(a)\right] \right) d\gamma_0^1(\mu) + \int_{S_2'} \frac{1-F(\mu)}{(1-\mu)f(\mu)} \left(\int_A \pi_S(a,\theta) \, d\left[\rho_0(T_0^1(\mu))(a) - \rho_0(\mu)(a)\right] \right) d\gamma_0^2(\mu) > 0$$

By Equation $\theta = 0$ we have $\sup_{\mu \in S_1'} \mathbf{E}_A\left[\pi_S(a,0)\rho_0(\mu)(a)\right] < \inf_{\mu \in S_2'} \mathbf{E}_A\left[\pi_S(a,0)\rho_0(\mu)(a)\right]$, combining this with the fact that $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is decreasing, we get the following lower bound for the right-hand side of the equation above

$$= \sup_{\mu'\in S_1'} \frac{1-F(\mu')}{(1-\mu')f(\mu')} \int_{S_1'} \left(\int_A \pi_S(a,\theta)\, d\left[\rho_0(T_0^2(\mu))(a) - \rho_0(\mu)(a)\right] \right) d\gamma_0^1(\mu)$$

$$+ \inf_{\mu'\in S_2'} \frac{1-F(\mu')}{(1-\mu')f(\mu')} \int_{S_2'} \left(\int_A \pi_S(a,\theta)\, d\left[\rho_0(T_0^1(\mu))(a) - \rho_0(\mu)(a)\right] \right) d\gamma_0^2(\mu)$$

By the definition of T_0^i we get the following

$$= \sup_{\mu'\in S_1'} \frac{1-F(\mu')}{(1-\mu')f(\mu')} \left(\int_{S_2'} \mathbf{E}_A\left[\pi_S(a,0)\rho_0(\mu)(a)\right]\, d\gamma_0^2(\mu) - \int_{S_1'} \mathbf{E}_A\left[\pi_S(a,0)\rho_0(\mu)(a)\right]\, d\gamma_0^1(\mu) \right)$$

$$+ \inf_{\mu'\in S_2'} \frac{1-F(\mu')}{(1-\mu')f(\mu')} \left(\int_{S_1'} \mathbf{E}_A\left[\pi_S(a,0)\rho_0(\mu)(a)\right]\, d\gamma_0^1(\mu) - \int_{S_2'} \mathbf{E}_A\left[\pi_S(a,0)\rho_0(\mu)(a)\right]\, d\gamma_0^2(\mu) \right)$$

The above is positive as by Equation θ= 0 we have $\sup_{\mu\in S_1'} \mathbf{E}_A\left[\pi_S(a,0)\rho_0(\mu)(a)\right] < \inf_{\mu\in S_2'} \mathbf{E}_A\left[\pi_S(a,0)\rho_0(\mu)(a)\right]$ and as $\sup_{\mu'\in S_1'} \frac{1-F(\mu')}{(1-\mu')f(\mu')} > \inf_{\mu'\in S_2'} \frac{1-F(\mu')}{(1-\mu')f(\mu')}$.

A similar argument holds for Equation θ= 1 after noting that $\frac{1-F(\mu)}{\mu f(\mu)}$ is decreasing. $\square$

2 Obedience Constraint

2.1 Iso-elastic distributions

Proposition 4. *If Assumption 1 holds then for any action a in A there exists a continuous distribution H_0^a with support $[\hat{\mu}(a), 1]$ such that Equation 17 holds with equality for all $\mu \geq \hat{\mu}(a)$.*

Proof. The iso-elastic distribution H_0^a is such that Equation 17 holds with equality for all $a' \geq a$. In particular, for all $a' \geq a$ the following holds

$$\int_\mu^1 \alpha_i(a') + \nu\alpha_s(a')\, dH_0^a(\nu) = \left(1 - \lim_{\mu'\uparrow 1} H_0^a(\mu')\right) \times \left(\ \alpha_i(a_1) + \alpha_s(a_1)\ \right)$$

Adding and subtracting $\int_\mu^1 \alpha_i(\hat{a}(\mu)) + \mu\alpha_s(\hat{a}(\mu))\, dH_0^a(\nu)$ to the left-hand side yields the following equality

$$\alpha_s(\hat{a}(\mu))\int_\mu^1 (\nu-\mu)\, dH_0^a(\nu) + \int_\mu^1 \alpha_i(\hat{a}(\mu)) + \mu\alpha_s(\hat{a}(\mu))\, dH_0^a(\nu) \\ = \left(1-\lim_{\mu'\uparrow 1} H_0^a(\mu')\right) \times (\ \alpha_i(a_1)+\alpha_s(a_1)\ ) \quad (48)$$

Equivalently, we have the following ordinary differential equation

$$\alpha_s(\hat{a}(\mu))\int_\mu^1 (\nu-\mu)\, dH_0^a(\nu) - (\alpha_i(\hat{a}(\mu)) + \mu\alpha_s(\hat{a}(\mu)))\frac{\partial}{\partial\mu}\int_\mu^1 (\nu-\mu)\, dH_0^a(\nu) \\ = \left(1-\lim_{\mu'\uparrow 1} H_0^a(\mu')\right) \times (\ \alpha_i(a_1)+\alpha_s(a_1)\ )$$

Dividing both sides by $-(\alpha_i(\hat{a}(\mu)) + \mu\alpha_s(\hat{a}(\mu)))$, which is positive by Assumption 1, we recover

$$\frac{d}{d\mu}\psi(\mu) + g_0(\mu)\psi(\mu) = g_1(\mu)\left(1-\lim_{\mu'\uparrow 1} H_0^a(\mu')\right)$$

Where

$$\psi(\mu) = \int_0^1 \max\{0, \nu-\mu\}\, dH_0^a(\nu)$$

and

$$g_0(\mu) = -\frac{\alpha_s(\hat{a}(\mu))}{\alpha_i(\hat{a}(\mu)) + \mu\alpha_s(\hat{a}(\mu))}, \quad g_1(\mu) = -\frac{\alpha_i(\hat{a}(1)) + \alpha_s(\hat{a}(1))}{\alpha_i(\hat{a}(\mu)) + \mu\alpha_s(\hat{a}(\mu))}$$

Using the integration factor

$$I(\mu) = \exp\left(-\int_\mu^1 g_0(\nu)\, d\nu\right)$$

We can express the differential equation as

$$\frac{d}{d\mu}\ (\psi(\mu)\times I(\mu)) = g_1(\mu)I(\mu)(1-\lim_{\mu'\uparrow 1} H_0^a(\mu'))$$

Integrating the expression on $[\mu, 1]$ gives the following solution

$$\psi(\mu) = -(1-\lim_{\mu'\uparrow 1} H_0^a(\mu')) \times \int_\mu^1 g_1(\nu)\frac{I(\nu)}{I(\mu)} d\nu \quad (49)$$

Differentiating and plugging in the initial condition $\frac{d}{d\mu}\psi(\mu)\Big|_{\mu=\hat{\mu}(a)} = -1$ we can solve Equation 18 to obtain the following

$$H_0^a(\nu) = \begin{cases} 0 & \nu \leq \hat{\mu}(a) \\ 1 - \frac{J(\nu)}{J(\hat{\mu}(a))} & \hat{\mu}(a) < \nu < 1 \\ 1 & \nu = 1 \end{cases}$$

For

$$J(\nu) = g_1(\nu) + g_0(\nu) \int_\nu^1 g_1(\mu') \frac{I(\mu')}{I(\nu)} d\mu'$$

The characteristic property of H_0^a is that it is supported on $[a_0, 1]$ (in other words Equation 5 holds) and that the receiver is indifferent between all actions in $[a, a_1]$ when $\operatorname{marg}_{\Delta\Theta} \beta(\cdot \mid a) = H_0^a$.

To show that $H_0^a(\nu)$ is a well-defined cumulative distribution function, we need to verify that it is non-decreasing in ν. By definition $H_0^a(\nu) = 1 - \frac{J(\nu)}{J(\hat{\mu}(a))}$ for $\hat{\mu}(a) < \nu < 1$. Differentiating $H_0^r(\nu)$ with respect to ν we get the following

$$\frac{\partial}{\partial \nu} H_0^a(\nu) = \frac{-1}{J(\hat{\mu}(a))} \frac{d}{d\nu} J(\nu)$$

To show that the above expression is positive by noting the following

$$\begin{aligned} \frac{d}{d\nu} J(\nu) &= \frac{d}{d\nu} g_1(\nu) + \left(\frac{d}{d\nu} g_0(\nu)\right) \int_\nu^1 g_1(\mu') \frac{I(\mu')}{I(\nu)} d\mu' - g_0^2(\nu) \int_\nu^1 g_1(\mu') \frac{I(\mu')}{I(\nu)} d\mu' - g_1(\nu) g_0(\nu) \\ &= \left(\frac{d}{d\nu} g_0(\nu) - g_0^2(\nu)\right) \int_\nu^1 g_1(\mu') \frac{I(\mu')}{I(\nu)} d\mu' + \frac{d}{d\nu} g_1(\nu) - g_1(\nu) g_0(\nu) \\ &= \frac{\left(\alpha_s(\hat{a}(\mu)) \frac{d}{d\mu} \alpha_i(\hat{a}(\mu)) - \alpha_i(\hat{a}(\mu)) \frac{d}{d\mu} \alpha_s(\hat{a}(\mu))\right) \int_\nu^1 g_1(\mu') \frac{I(\mu')}{I(\nu)} d\mu'}{(\alpha_i(\hat{a}(\mu)) + \mu \alpha_s(\hat{a}(\mu)))^2} \\ &\qquad + \frac{(\alpha_i(\hat{a}(1)) + \alpha_s(\hat{a}(1)))(\frac{d}{d\mu} \alpha_i(\hat{a}(\mu)) + \mu \frac{d}{d\mu} \alpha_s(\hat{a}(\mu)))}{(\alpha_i(\hat{a}(\mu)) + \mu \alpha_s(\hat{a}(\mu)))^2} \end{aligned}$$

The expression above is positive under Assumption 1. To see this, consider the following inequalities

$$\alpha_s(\hat{a}(\mu)) \frac{d}{d\mu} \alpha_i(\hat{a}(\mu)) - \alpha_i(\hat{a}(\mu)) \frac{d}{d\mu} \alpha_s(\hat{a}(\mu))$$

$$= \alpha_i' \cdot \underbrace{\left(-\zeta \circ \alpha_i \;+\; \alpha_i \;\cdot\; \zeta' \circ \alpha_i \right)}_{\text{sub-homogeneous } \zeta \implies \leq 0} \cdot \hat{a}' \leq 0$$

and

$$\begin{aligned} &\frac{d}{d\mu}\alpha_i(\hat{a}(\mu)) + \mu \frac{d}{d\mu}\alpha_s(\hat{a}(\mu)) \\ &= \alpha_i' \cdot \underbrace{\left(1 - \mu\zeta' \circ \alpha_i \right)}_{\text{1-Lipschitz } \zeta \implies \geq 0} \geq 0 \end{aligned}$$

Thus, we have that J is increasing

$$\frac{d}{d\nu}J(\nu) \geq 0$$

As $J(1) = g_1(1) = -1$, and as J is increasing we get that $J(\hat{\mu}(a)) < 0$ and thus $\frac{\partial}{\partial\nu}H_0^a(\nu) > 0$. □

By Equation 49 the mean of H_0^r is given by

$$\begin{aligned} \mathbf{E}H_0^r &= \hat{\mu}(a) - (1 - \lim_{\mu' \uparrow 1} H_0^a(\mu')) \times \int_{\hat{\mu}(a)}^1 g_1(\nu)\frac{I(\nu)}{I(\hat{\mu}(a))}d\nu \\ &= \hat{\mu}(a) + \frac{1}{J(\hat{\mu}(a))}\int_{\hat{\mu}(a)}^1 g_1(\nu)\frac{I(\nu)}{I(\hat{\mu}(a))}d\nu \\ \implies \frac{d}{da}\mathbf{E}H_0^a &= \left[1 + \frac{d}{d\hat{\mu}(a)}\left(\frac{1}{J(\hat{\mu}(a))}\int_{\hat{\mu}(a)}^1 g_1(\nu)\frac{I(\nu)}{I(\hat{\mu}(r))}d\nu\right)\right]\frac{d}{dr}\hat{\mu}(a) \\ &= \int_{\hat{\mu}(a)}^1 g_1(\nu)\frac{I(\nu)}{I(\hat{\mu}(a))}d\nu \times \frac{d}{d\hat{\mu}(a)}\frac{1}{J(\hat{\mu}(a))} \times \frac{d}{da}\hat{\mu}(a) \end{aligned}$$

As $\hat{\mu}(a)$ is increasing in a, $J(\mu)$ is increasing in μ and $g_1 \leq 0$, we get that $\mathbf{E}H_0^a$ is increasing in a.

2.2 Extended Family of Distributions

Additionally, we can define a family of distributions H_x^a with expectation ranging in $[\hat{\mu}(a), \mathbf{E}H_0^a]$ and such that the receiver is indifferent between all actions a' for which $\hat{\mu}(a')$ is in the support of H^a_x.

$$H^a_x(\nu) = \begin{cases} 0 & \nu < \hat{\mu}(a) \\ x & \hat{\mu}(a) \leq \nu \leq \widetilde{\mu}(x) \\ x + (1-x)H_0^{\hat{a}(\widetilde{\mu}(x))}(\nu) & \widetilde{\mu}(x) < \nu < 1 \\ 1 & \nu = 1 \end{cases} \tag{50}$$

Let $x_r = \frac{\alpha_i(\hat{a}(1))+\alpha_s(\hat{a}(1))-\alpha_i(a)-\alpha_s(a)}{\alpha_i(\hat{a}(1))+\alpha_s(\hat{a}(1))-(1-\hat{\mu}(a))\alpha_s(a)}$, which is in $[0,1]$ by Assumption 1.

Then $\widetilde{\mu}(x) = 1$ for $x > x_r$ and $\widetilde{\mu}(x)$ is the solution to the following equality when $x \leq x_r$

$$\alpha_i(a) + \left(x\hat{\mu}(a) + (1-x)\mathbf{E}H_0^{\hat{a}(\widetilde{\mu}(x))}\right)\alpha_s(a) = \\ (1-x)(\alpha_i(\hat{a}(1)) - \alpha_s(\hat{a}(1)))\left(1 - \lim_{\mu'\uparrow 1} H_0^{\hat{a}(\widetilde{\mu}(x))}(\mu')\right) \tag{51}$$

The left-hand side above is decreasing in $\widetilde{\mu}(x)$ as $\mathbf{E}H^a_0$ increases in a, and the right-hand side is non-decreasing in $\widetilde{\mu}(x)$ as $\lim_{\mu'\uparrow 1} H^a_0(\mu') = 1 + \frac{1}{J(\hat{\mu}(a'))}$ and as J is non-decreasing. This implies that $\widetilde{\mu}(x)$ is well defined.

2.3 Feasible Mean

Proof of Lemma 2. Assume for contradiction that G is supported on $[\hat{\mu}(a), 1]$, satisfies Equation 21 and has a mean $m > \mathbf{E}H^a_0$. By Equation 21 we must have the following

$$(1 - \lim_{\mu'\uparrow 1} G(\mu'))(\alpha_i(\hat{a}(1)) + \alpha_s(\hat{a}(1))) \leq \alpha_i(a) + m\alpha_s(a)$$

By assumption $m > \mathbf{E}H^r_0$ and $\alpha_s(a) \leq 0$ thus we get

$$(1 - \lim_{\mu'\uparrow 1} G(\mu'))(\alpha_i(\hat{a}(1)) + \alpha_s(\hat{a}(1))) \leq \alpha_i(a) + \alpha_s(a)\mathbf{E}H^a_0$$

As the obedience constraint holds with equality for H^a_0 for all μ in $[\hat{\mu}(a), 1]$, we get the following

$$\alpha_i(a) + \alpha_s(a)\mathbf{E}H^a_0 = (1 - \lim_{\mu'\uparrow 1} H^a_0(\mu'))(\alpha_i(\hat{a}(1)) + \alpha_s(\hat{a}(1)))$$

Combining the two equations above yields

$$\lim_{\mu'\uparrow 1} G(\mu') > \lim_{\mu'\uparrow 1} H_0^a(\mu')$$

As $\mathbf{E}G > \mathbf{E}H_0^a$ the CDF G can not be everywhere above H_0^a. By continuity of H_0^a and right continuity of G we get that there exists some $\hat{\mu}(a) < \mu_0 < 1$ such that $\lim_{\mu'\uparrow\mu_0} G(\mu') \le H_0^a(\mu_0) \le G(\mu_0)$ and $H_0^a(\mu) \le G(\mu)$ for all $\mu > \mu_0$. Evaluating the lender's payoff from setting a rate $\hat{a}(\mu_0)$ against the distribution G we get

$$\alpha_i(\hat{a}(\mu_0))(1 - \lim_{\mu'\uparrow\mu_0} G(\mu')) + \alpha_s(\hat{a}(\mu_0)) \int_{[\mu_0,1]} \nu dG(\nu)$$

$$\geq$$

$$\alpha_i(\hat{a}(\mu_0))(1 - \lim_{\mu'\uparrow\mu_0} G(\mu')) + \alpha_s(\hat{a}(\mu_0)) \int_{[\mu_0,1]} \nu dG(\nu) - (\alpha_i(\hat{a}(\mu_0)) + \mu_0\alpha_s(\hat{a}(\mu_0)))(H_0^r(\mu_0) - G(\mu_0))$$

As H_0^a is (weakly) below G for all $\mu > \mu_0$, we get that the above expression is bounded below by the following

$$\alpha_i(\hat{a}(\mu_0))(1 - H_0^a(\mu_0)) + \alpha_s(\hat{a}(\mu_0)) \int_{\mu_0}^1 \nu dH_0^r(\nu)$$

By binding obedience constraint for H_0^a, the above equals

$$\alpha_i(a) + \alpha_s(a)\mathbf{E}H_0^a$$

Thus, if $m > \mathbf{E}H_0^a$, setting a rate $\hat{a}(\mu_0)$ yields greater payoff to the lender than setting a rate a, hence contradicting Equation 21. □

3 Proofs for Appendix C

Proof of Lemma 3. Consider any q_1 for which there is an interval of types, $[\mu_0, \overline{\mu}]$ such that $\max\left\{\frac{(1-\mu)K(\mu)}{1-p}, \frac{K(\mu)-p}{1-p}\right\} < q_1(\mu) < \min\left\{1, \frac{K(\mu)}{1-p}\right\}$. Pick $\delta > 0$ such that $\overline{\mu} - \mu_0 \geq 2\delta$. We can improve the revenue by increasing q_1 on $[\overline{\mu} - \delta, \overline{\mu})$ by some $\varepsilon_1 > 0$ and by reducing q_1 on $[\mu_0, \mu_0 + \delta)$ by $\varepsilon_0 > 0$ where

$$\varepsilon_1 = \varepsilon_0 \frac{\int_{\mu_0}^{\mu_0+\delta} \left(1 - p - \mu\left(1 - \frac{p}{m}\right)\right) dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} \left(1 - p - \mu\left(1 - \frac{p}{m}\right)\right) dF(\mu)}$$

ensuring that the target mean, i.e. Equation 37 is maintained. This adjustment doesn't violate the constraint in Equation 26 as it weakly decreases the size of any point mass at p without changing the target mean.

The revenue from the new allocation is greater than the old allocation if the following holds:

$$\frac{\int_{\mu_0}^{\mu_0+\delta}\left(1-p-\mu\left(1-\frac{p}{m}\right)\right)dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}}\left(1-p-\mu\left(1-\frac{p}{m}\right)\right)dF(\mu)} \geq \frac{\int_{\mu_0}^{\mu_0+\delta}\mu\left(1-\frac{p}{m}\right)dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}}\mu\left(1-\frac{p}{m}\right)dF(\mu)}$$

$$\iff \frac{\int_{\mu_0}^{\mu_0+\delta}(1-p)\,dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}}(1-p)\,dF(\mu)} \geq \frac{\int_{\mu_0}^{\mu_0+\delta}\mu\left(1-\frac{p}{m}\right)dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}}\mu\left(1-\frac{p}{m}\right)dF(\mu)}$$

$$\iff \frac{\int_{\mu_0}^{\mu_0+\delta}dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}}dF(\mu)} \geq \frac{\int_{\mu_0}^{\mu_0+\delta}\mu dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}}\mu dF(\mu)}$$

The last inequality follows from our choice of δ.

Now consider q_1 such that there are intervals $I_1 < I_2$ where $q_1(\mu) = \max\left\{\frac{(1-\mu)K(\mu)}{1-p}, \frac{K(\mu)-p}{1-p}\right\}$ for $\mu \in I_2$ and $q_1(\mu) = \min\left\{1, \frac{K(\mu)}{1-p}\right\}$ for $\mu \in I_1$. We construct an improvement similar to the above by slightly increasing q_1 on I_2 while reducing q_1 on I_1 to keep the mean constraint binding. Finally, note that this improvement introduces more slack to Equation 26 . The revenue of this improvement is greater than the old allocation if the following holds:

$$\frac{\int_{I_1}dF(\mu)}{\int_{I_2}dF(\mu)} \geq \frac{\int_{I_1}\mu dF(\mu)}{\int_{I_2}\mu dF(\mu)}$$

This is implied by $\frac{\inf(I_2)}{\sup(I_1)} \geq 1$. □

Proof of Lemma 4. Consider a feasible allocation with target mean m and mass size x. Suppose there is a non-empty interval $[\underline{\mu}, \mu_0)$ consiting of types μ for which $q_1(\mu) = \frac{(1-\mu)K(\mu)}{1-p} > 0$. Then $x > 0$ and $m \leq \mathbf{E}H_0^p$. Consider a new allocation which reduces $K(\mu)$ slightly at all points in $[\underline{\mu}, \mu_0)$ while keeping q_1 unchanged. The new allocation increases the mean slightly (by Equation 25) to say $m' > m$ and eliminates any mass point (Equation 6 will be slack). For m' close enough to

m the new allocation is feasible for the relaxed problem with target mean m' and mass $x = 0$. That is, Equation 25 and Equation 26 are satisfied for m' and $x = 0$ because $\mathbf{E}H_0^p > \mathbf{E}H_x^p$.

By inspection of the objective in Equation 38 this is an improvement. Therefore a solution to the relaxed problem must have $K(\mu) = 0$ for all $\mu < \mu_0$. $\square$

Proof of Lemma 5. We first show that any candidate solution can be weakly improved by one that has $q_0(\mu) \in \{0, 1\}$ for all $\mu \in [\overline{\mu}, 1]$. If the candidate allocation does not already have that property then we construct a new allocation q' such that on $q'(\mu) = q(\mu)$ for $\mu \in [\overline{\mu}, 1]^C$. On the interval $[\overline{\mu}, \overline{\mu} + \varepsilon_0)$ let $q_0'(\mu) = q_0(\overline{\mu})$, on the interval $[1 - \varepsilon_1, 1]$ let $q_0'(\mu) = 1$, and $q_0'(\mu) = q_0(\mu)$ for $\mu \in [\overline{\mu} + \varepsilon_0, 1 - \varepsilon_1)$. Where

$$\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon_0} (1-\mu)(q_0(\mu) - q_0(\overline{\mu}))dF(\mu) - \int_{1-\varepsilon_1}^{1} (1-\mu)(1-q_0(\mu))dF(\mu) = 0$$

As long as $\overline{\mu} + \varepsilon_0 \leq 1 - \varepsilon_1$, the allocation q' is well-defined and has the same target mean as q. Consider the difference in revenue between q and q':

$$\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon_0} \left(1 - \mu + \frac{1-F(\mu)}{f(\mu)}\right)(q_0(\mu) - q_0(\overline{\mu}))dF(\mu) - \int_{1-\varepsilon_1}^{1} \left(1 - \mu + \frac{1-F(\mu)}{f(\mu)}\right)(1 - q_0(\mu))dF(\mu)$$

$$= \int_{\overline{\mu}}^{\overline{\mu}+\varepsilon_0} \frac{1-F(\mu)}{f(\mu)}(q_0(\mu) - q_0(\overline{\mu}))dF(\mu) - \int_{1-\varepsilon_1}^{1} \frac{1-F(\mu)}{f(\mu)}(1-q_0(\mu))dF(\mu)$$

When $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing the change in revenue is positive as

$$\int_{1-\varepsilon_1}^{1} \frac{1-F(\mu)}{f(\mu)}(1-q_0(\mu))dF(\mu) \leq \frac{1-F(1-\varepsilon_1)}{\varepsilon_1 f(1-\varepsilon_1)} \int_{1-\varepsilon_1}^{1} (1-\mu)(1-q_0(\mu))dF(\mu)$$

$$\implies \int_{1-\varepsilon_1}^{1} \frac{1-F(\mu)}{f(\mu)}(1-q_0(\mu))dF(\mu) \leq \frac{1-F(1-\varepsilon_1)}{\varepsilon_1 f(1-\varepsilon_1)} \int_{\overline{\mu}}^{\overline{\mu}+\varepsilon_0} (1-\mu)(q_0(\mu) - q_0(\overline{\mu}))dF(\mu)$$

$$\implies \int_{1-\varepsilon_1}^{1} \frac{1-F(\mu)}{f(\mu)}(1-q_0(\mu))dF(\mu) \leq \int_{\overline{\mu}}^{\overline{\mu}+\varepsilon_0} \frac{1-F(\mu)}{f(\mu)}(q_0(\mu) - q_0(\overline{\mu}))dF(\mu)$$

The first implication follows from the choice of $\varepsilon_0, \varepsilon_1$. Thus any candidate allocation can be weakly improved by one for which there exists $\mu' \in [\overline{\mu}, 1]$ such that $q_0(\mu) = 0$ on $[\overline{\mu}, \mu')$ and $q_0(\mu) = 1$ on $[\mu', 1]$. Note that the latter implies item 1 in the statement of the Lemma in view of Equation 41.

In particular, by Equation 41, the improved allocation has $q_0(\mu) = 1$ for $\mu \in [\mu_0, \mu_1)$ and $q_0(\mu) = 0$ for $\mu \in [\mu_1, \mu')$. Next we claim that any such allocation for which μ_1 is strictly between μ_0 and μ' can be improved by one which satisfies item 2 in the statement of the Lemma. In other words, one for which μ_1 equals either μ_0 or μ'.

Assume for contradiction that the q is such that $\mu_0 < \mu_1 < \mu'$. We construct a new allocation q' such that on $q'(\mu) = q(\mu)$ for $\mu \in [\mu_0, \mu')^C$. On the interval $[\mu_0, \mu_1)$ we require $q'_0(\mu) = q_0(\mu) - \varepsilon_0$, on the interval $[\mu_1, \mu')$ we require $q'_0(\mu) = q_0(\mu) + \varepsilon_1$. Where

$$\varepsilon_1 = \varepsilon_0 \frac{\int_{\mu_0}^{\mu_1} (1-\mu) dF(\mu)}{\int_{\mu_1}^{\mu'} (1-\mu) dF(\mu)}$$

For small enough $\varepsilon_0 > 0$, the allocation q' is feasible with the same target mean as q. We argue that under the condition of the proposition, this q' achieves a weakly higher revenue. To see this, consider the difference in revenue between q and q':

$$\varepsilon_0 \int_{\mu_0}^{\mu_1} \left[1 - \mu + \frac{1-F(\mu)}{f(\mu)}\right] dF(\mu) - \varepsilon_1 \int_{\mu_1}^{\mu'} \left[1 - \mu + \frac{1-F(\mu)}{f(\mu)}\right] dF(\mu)$$

Similar to Proposition Proposition 1, when $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing the change in revenue is positive as

$$\frac{\int_{\mu_0}^{\mu_1} \frac{1-F(\mu)}{f(\mu)} dF(\mu)}{\int_{\mu_1}^{\mu'} \frac{1-F(\mu)}{f(\mu)} dF(\mu)} > \frac{\int_{\mu_0}^{\mu_1} (1-\mu) dF(\mu)}{\int_{\mu_1}^{\mu'} (1-\mu) dF(\mu)}.$$

□

4 Optimal Mean– Platform

Proposition 2 characterizes the structure of the optimal mechanism for the relaxed problem. Using this structure, we can show that the optimal target mean for a given price p is $\mathbf{E}H_0^p$.

Lemma 12. *If $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing then the optimal mean in Equation 25 for a solution to the relaxed problem with price p is $m = \mathbf{E}H_0^p$.*

Proof. Consider for contradiction that q is of one of the forms identified in Proposition 2 but the corresponding target mean $m < \mathbf{E}H_0^p$.

Case I: Let q be of the first form

$$q(\mu) = \begin{cases} (0,0) & \text{if} \quad \mu \le \mu_0 \\ \left(\frac{p}{1-p}\frac{1-\mu_0}{\mu_0}, 1\right) & \text{if} \quad \mu \in [\mu_0, \mu') \\ (1,1) & \quad\;\; \text{otherwise} \end{cases}$$

If $\mu_0 < \mu'$ then we can construct a profitable deviation by choosing $\delta^*, \delta > 0$ and by perturbing the allocation to $(0,0)$ on $[\mu_0, \mu_0 + \delta^*)$ and to $(1,1)$ on $[\mu' - \delta, \mu')$. We can choose δ, δ^* such that the new mean of the allocation is $m' \le \mathbf{E}H_0^p$ and the following holds

$$\int_{\mu_0}^{\mu_0+\delta^*} \mu \frac{p}{1-p}\frac{1-\mu_0}{\mu_0}\, dF(\mu) = \int_{\mu'-\delta}^{\mu'} \mu \left(1 - \frac{p}{1-p}\frac{1-\mu_0}{\mu_0}\right) dF(\mu)$$

As $\frac{1-F(\mu)}{\mu f(\mu)}$ is non-increasing, this perturbation reduces the information rent and increases the gross surplus.

If $\mu_0 = \mu'$ then we can create a profitable perturbation by changing the allocation to $(1,0)$ on $[\mu', \mu' + \delta)$ for small enough $\delta > 0$.

Case II: Let q be of the second form

$$q(\mu) = \begin{cases} (0,0) & \text{if} \quad \mu \le \mu_0 \\ (1,0) & \text{if} \quad \mu \in [\mu_0, \mu') \\ (1,1) & \quad\;\; \text{otherwise} \end{cases}$$

If $\mu_0 = \mu'$, we can repeat the previous argument.

If $\mu_0 < \mu'$ then we can construct a profitable deviation by choosing $\delta > 0$ and by perturbing the allocation to $(1,0)$ on $[\mu', \mu' + \delta^*)$. This is feasible for a small enough δ as $m < \mathbf{E}H_0^p \le 1$ implies $\mu' < 1$. □

5 Uniform Prior– Theorem 3

So far, we have characterized the qualitative structure of the optimal relaxed mechanism under some distributional constraints. In this section, we demonstrate the power of our characterization by deriving the optimal primal mechanism for a uniform prior. Consider $F \sim \text{Unif}[0,1]$, this satisfies all conditions of Proposition 2, thus the optimal solution to the relaxed problem can be expressed as one of the mechanisms identified in the proposition. Moreover, by Lemma 12 it suffices to consider Equation 25 where $m = \mathbf{E}H_0^p = p(1 - \ln p)$. First, we look at the mechanism of the form

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu < \mu_0 \\ (1,0) & \mu \in [\mu_0, \overline{\mu}) \\ (1,1) & \mu > \overline{\mu} \end{cases}$$

The mean constraint binding with no point mass at p implies that the relaxed problem can be written as

$$\max_{\mu_0, \overline{\mu}} \int_{\mu_0}^{1} (1-p)(2\mu - 1) d\mu - 2p \int_{\overline{\mu}}^{1} (1-\mu) d\mu$$

s.t.

$$0 \leq \mu_0 \leq \overline{\mu} \leq 1$$

$$\int_{\mu_0}^{1} \mu d\mu = \frac{\mathbf{E}H_0^p}{1 - \mathbf{E}H_0^p} \int_{\overline{\mu}}^{1} (1-\mu) d\mu$$

The objective can be rewritten as

$$\int_{\mu_0}^{1} (1-p)(2\mu - 1) d\mu - 2p \frac{1 - \mathbf{E}H_0^p}{\mathbf{E}H_0^p} \int_{\mu_0}^{1} \mu d\mu$$

Note that $\mu_0 = \mu^*(p) := \frac{1-p}{2\left(1 - \frac{p}{\mathbf{E}H_0^p}\right)}$ is the pointwise maximizer of the objective. Let $p_1 := \text{argmax}\left\{ p \in [0,1] \,\middle|\, \mu^*(p) \geq t(\mu^*(p)) \right\}$, where $t(\mu^*) = 1 - \sqrt{\frac{(1 - \mathbf{E}H_0^p)}{\mathbf{E}H_0^p}(1 - (\mu^*)^2)}$.

For $p \geq p_1$, the point-wise optimal mechanism is feasible for the relaxed problem. Moreover, the revenue is given by

$$\int_{\mu^*(p)}^{1} \left[2\mu \left(1 - \frac{p}{\mathbf{E}H_0^p} \right) - (1-p) \right] d\mu$$

The above is decreasing in p, the derivative of the revenue is

$$\int_{\mu^*(p)}^{1} 1 - \frac{2\mu}{p}\frac{1}{(1-\ln(p))^2} d\mu$$

This is negative as $\frac{1-3\ln(p)-p(1-\ln(p))}{-2p\ln(p)(1-\ln(p))^2} > 1$ for all $p \in [0,1]$.

For $p < p_1$, the point-wise optimal mechanism doesn't satisfy the ex-post participation constraint. The ex-post participation fails for $p < p_1$ as $\mu_1 < \mu^*(p)$. Note that the revenue decreases if $q_1(\mu)$ increases for $\mu < \mu^*(p)$. Thus, if $p < p_1$ then the optimal solution to the relaxed problem, restricted to mechanisms above, involves thresholds $\mu_0 = \overline{\mu}$. By the mean constraint, the threshold value μ_0 is a root of the following quadratic equation

$$(1-\overline{\mu})^2 = \frac{1-\mathbf{E}H_0^p}{\mathbf{E}H_0^p}(1-\overline{\mu}^2)$$

This has a single interior solution, $\mu_0 = \overline{\mu} = 2\mathbf{E}H_0^p - 1$, the revenue is thus given by

$$\int_{2\mathbf{E}H_0^p-1}^{1} \left[2\mu\left(1-\frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right] d\mu$$

The above is negative if $p_1 > p \geq p_0 := \operatorname{argmax}\left\{p \in [0,1] \,\middle|\, \frac{1+p}{2} \geq p(1-\ln p)\right\}$, thus the optimal thresholds are such that is if $p \in [p_0, p_1]$ then $\mu_0 = \overline{\mu} = 2\mathbf{E}H_0^p - 1$ and if $p < p_0$ then $\mu_0 = 1$. Thus, to find the optimal mechanism in the first class of mechanisms from the Proposition 2, we will maximize the following objective with respect to $p \in [p_0, p_1]$.

$$R(p) := \int_{2\mathbf{E}H_0^p-1}^{1} \left[2\mu\left(1-\frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right] d\mu = 2\left[1-\mathbf{E}H_0^p\right]\left[2\mathbf{E}H_0^p - (1+p)\right]$$

This function is concave in the domain and achieves an interior maximum. Unfortunately, the first order condition is a transcendental equation, so we rely on numerical methods to calculate the optimum. The optimal revenue for the relaxed problem among this class of mechanisms is $\simeq 0.0646$, which is achieved by $p \simeq 0.4364$, and $\mu_0 = \overline{\mu} \simeq 0.596$.

Now we will describe the optimal mechanism for the second class of mechanisms identified in Proposition 2;

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu < \mu_0 \\ \left(\frac{p(1-\mu_0)}{(1-p)\mu_0}, 1\right) & \mu \in [\mu_0, \overline{\mu}) \\ (1,1) & \mu > \overline{\mu} \end{cases}$$

We only need to optimize over these mechanisms for price $p < p_1$, as for other prices the mechanism discussed above maximizes the revenue point-wise. We solve the following problem

$$\max_{\mu_0, \overline{\mu}} \int_{\mu_0}^{\overline{\mu}} (1-p)\frac{p(1-\mu_0)}{(1-p)\mu_0}(2\mu - 1)d\mu + \int_{\overline{\mu}}^{1}(1-p)(2\mu-1)d\mu - 2p\int_{\mu_0}^{1}(1-\mu)d\mu$$

s.t.

$$p \leq \mu_0 \leq \overline{\mu} \leq 1$$

$$\int_{\mu_0}^{\overline{\mu}} \mu \frac{p(1-\mu_0)}{(1-p)\mu_0} d\mu + \int_{\mu_0}^{1} \mu d\mu = \frac{\mathbf{E}H_0^p}{1-\mathbf{E}H_0^p}\int_{\mu_0}^{1}(1-\mu)d\mu$$

The objective can be rewritten as

$$\int_{\mu_0}^{\overline{\mu}} \frac{p(1-\mu_0)}{(1-p)\mu_0}\left[2\mu\left(1 - \frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right]d\mu + \int_{\overline{\mu}}^{1}\left[2\mu\left(1-\frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right]d\mu$$

From the obedience we can express $\overline{\mu} = h(\mu_0) := \sqrt{\mu_0^2 + \frac{(1-p)\mu_0(1-\mu_0)}{(\mu_0-p)(1-\mathbf{E}H_0^p)}(1+\mu_0 - 2\mathbf{E}H_0^p)}$ For feasibility we require $p \leq \mu_0 \leq \overline{\mu}$, thus feasibility can be restated as $\max\{p, 2\mathbf{E}H_0^p - 1\} \leq \mu_0$. The optimization problem can be restated

$$\int_{\mu_0}^{h(\mu_0)} \frac{p(1-\mu_0)}{(1-p)\mu_0}\left[2\mu\left(1 - \frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right]d\mu + \int_{h(\mu_0)}^{1}\left[2\mu\left(1-\frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right]d\mu$$

s.t.

$$\max\{p, 2\mathbf{E}H_0^p - 1\} \leq \mu_0 \leq 1$$

This is a well-defined two-variable unconstrained optimization on a compact set. In particular, we can numerically derive the relevant features of the mechanism at the optimum. The optimal revenue for the relaxed problem among this class of mechanisms is $\simeq 0.0651$, which is achieved by $p^* \simeq 0.417$, $\mu_0 \simeq 0.578$, and $\overline{\mu} \simeq 0.629$. Thus optimal relaxed mechanism with revenue-maximizing price p^* is given by

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu < 0.578 \\ (0.522, 1) & \mu \in [0.578, 0.629) \\ (1,1) & \mu > 0.629 \end{cases}$$

The CDF for the receiver's second-order beliefs for the above mechanism is given by

$$\operatorname{marg}_{\Delta(\theta)} \beta(\mu \mid p = 0.417) = \begin{cases} 0 & \mu < 0.417 \\ w^{-1} \int_{0.578}^{\frac{\mu}{\mu+0.522(1-\mu)}} (0.522s + (1-s))ds & \mu \in [0.417, 0.47) \\ w^{-1} \int_{0.578}^{0.629} (0.522s + (1-s))ds & \mu \in [0.47, 0.629) \\ \frac{\left[\int_{0.578}^{0.629}(0.522s+(1-s))ds + \int_{0.629}^{\mu}(s+(1-s))ds\right]}{w} & \mu \geq 0.629 \end{cases}$$

Where $w = \int_{0.578}^{0.629} 0.522s ds + \int_{0.629}^{1} ds + \int_{0.578}^{0.629} (1-s)\, ds$. As the mechanism solves the relaxed problem, $\operatorname{marg}_{\Delta(\theta)} \beta(\cdot \mid p = 0.417)$ has the same expectation as $H_0^{0.417}$. We claim that $H_0^{0.417} \succeq_{\text{mps}} \operatorname{marg}_{\Delta(\theta)} \beta(\cdot \mid p = 0.417)$, this follows from the fact that the distributions have the same support and that $\operatorname{marg}_{\Delta(\theta)} \beta(\cdot \mid p = 0.417)$ crosses $H_0^{0.417}$ exactly once from below.

6 Proofs for Appendix D

Proof of Lemma 6. Let $m \in (\hat{\mu}(r), \mathbf{E}H_0^r]$ for $r < \hat{r}(1)$, the boundary case follow from setting $\mu^K = 1$ when $m = 1$ and setting $\mu^K = 0$, $\mu_0 = 1$ when $m = \hat{\mu}(r)$.

Consider any q_1 for which there is an interval of types in $[\mu', \mu''] \subset [0, \mu_0]$ such that $\frac{K(\mu)}{u_1(r)} < q_1(\mu) < \frac{K(\mu)}{u_1(r) - \alpha_r(\mu) u_0(r)}$. Pick $\delta > 0$ such that $\mu'' - \mu' \geq 2\delta$. We can improve the revenue by increasing q_1 on $[\mu'' - \delta, \mu'')$ by some $\varepsilon_1 > 0$ and by reducing q_1 on $[\mu', \mu' + \delta)$ by $\varepsilon_0 > 0$ where

$$\varepsilon_1 = \varepsilon_0 \frac{\int_{\mu'}^{\mu'+\delta} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)}$$

ensuring that the target mean, i.e. Equation 45 is maintained. This is possible as $(1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)$ is positive on $[0, \mu_0]$. We can choose ε_0 small enough such that the adjustment doesn't violate the constraint in Equation 29 as it weakly decreases the size of any point mass at $\hat{\mu}(r)$ without changing the target mean.

The revenue from the new allocation is greater than the old allocation if the following holds:

$$\frac{\int_{\mu'}^{\mu'+\delta} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta} \mu \left(u_1(r) + u_0(r)\frac{1-m}{m} - \frac{R_0}{m}\right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu \left(u_1(r) + u_0(r)\frac{1-m}{m} - \frac{R_0}{m}\right) dF(\mu)}$$

$$\iff \frac{\int_{\mu'}^{\mu'+\delta} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta} \mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu dF(\mu)}$$

$$\iff \frac{\int_{\mu'}^{\mu'+\delta} u_1(r) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} u_1(r) dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta} \mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu dF(\mu)}$$

$$\iff \frac{\int_{\mu'}^{\mu'+\delta} dF(\mu)}{\int_{\mu''-\delta}^{\mu''} dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta} \mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu dF(\mu)}$$

The first implication follows as $m \geq \hat{\mu}(r)$ implies that $u_1(r) + u_0(r)\frac{1-m}{m} - \frac{R_0}{m} \geq 0$. The second implication holds as $\mu \leq \mu_0 < \frac{u_1(r)}{u_1(r) + \frac{1-m}{m}u_0(\mu)}$ which implies that $(1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r) > 0$. The last inequality follows from our choice of δ.

Now consider q_1 such that there are intervals $I_1 < I_2 \subset [0, \mu_0]$ where $q_1(\mu) = \frac{K(\mu)}{u_1(r) - \alpha_r(\mu)u_0(r)}$ for $\mu \in I_1$ and $q_1(\mu) = \frac{K(\mu)}{u_1(r)}$ for $\mu \in I_2$. We construct an improvement similar to the above by slightly increasing q_1 on I_2 while reducing q_1 on I_1 to keep the mean constraint binding. Finally, note that this improvement introduces more slack to Equation 29. The revenue of this improvement is greater than the old allocation if the following holds:

$$\frac{\int_{I_1} dF(\mu)}{\int_{I_2} dF(\mu)} \geq \frac{\int_{I_1} \mu dF(\mu)}{\int_{I_2} \mu dF(\mu)}$$

This is implied by $\frac{\inf(I_2)}{\sup(I_1)} \geq 1$.

We can repeat the above arguments to establish that q_1 can not be such that $\frac{K(\mu)}{u_1(r)} < q_1(\mu) < \min\left\{1, \frac{K(\mu)+u_0(r)}{u_1(r)}\right\}$ for $\mu_0 \leq \mu \leq \frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}$. Moreover, there are no intervals $I_1 < I_2 \subset \left[\mu_0, \frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}\right]$ such that $q_1(\mu) = \min\left\{1, \frac{K(\mu)+u_0(r)}{u_1(r)}\right\}$ for $\mu \in I_1$ and $q_1(\mu) = \frac{K(\mu)}{u_1(r)}$ for $\mu \in I_2$.

To complete the proof consider $\mu > \frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}$. Assume for contradiction that there is an interval of types in $[\mu', \mu''] \subset [0, \mu_0]$ such that $\frac{K(\mu)}{u_1(r)} < q_1(\mu) < \min\left\{1, \frac{K(\mu)+u_0(r)}{u_1(r)}\right\}$. Pick $\delta > 0$ such that $\mu'' - \mu' \geq 2\delta$. We can improve the revenue by decreasing q_1 on $[\mu'' - \delta, \mu'')$ by some $\varepsilon_1 > 0$ and by increasing q_1 on $[\mu', \mu' + \delta)$ by $\varepsilon_0 > 0$ where

$$\varepsilon_1 = \varepsilon_0 \frac{\int_{\mu'}^{\mu'+\delta} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)}$$

Again, this adjustment can be made while maintaining Equation 45 and Equation 29.

The revenue from the new allocation is greater than the old allocation if the following holds:

$$-\frac{\int_{\mu'}^{\mu'+\delta} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)} \geq -\frac{\int_{\mu'}^{\mu'+\delta} \mu\left(u_1(r) + u_0(r)\frac{1-m}{m} - \frac{R_0}{m}\right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu\left(u_1(r) + u_0(r)\frac{1-m}{m} - \frac{R_0}{m}\right) dF(\mu)}$$

$$\iff -\frac{\int_{\mu'}^{\mu'+\delta} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)} \geq -\frac{\int_{\mu'}^{\mu'+\delta} \mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu dF(\mu)}$$

$$\iff \frac{\int_{\mu'}^{\mu'+\delta} u_1(r)dF(\mu)}{\int_{\mu''-\delta}^{\mu''} u_1(r)dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta} \mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu dF(\mu)}$$

$$\iff \frac{\int_{\mu'}^{\mu'+\delta} dF(\mu)}{\int_{\mu''-\delta}^{\mu''} dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta} \mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu dF(\mu)}$$

The second implication holds as $\mu \geq \frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}$ which implies that $(1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r) < 0$. The last inequality follows from our choice of δ.

Similarly, we can show that there are no intervals $I_1 < I_2 \subset \left[\frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}, 1\right]$ such that $q_1(\mu) = \min\left\{1, \frac{K(\mu)+u_0(r)}{u_1(r)}\right\}$ for $\mu \in I_2$ and $q_1(\mu) = \frac{K(\mu)}{u_1(r)}$ for $\mu \in I_1$. ☐

Proof of Lemma 7. Consider $r < \hat{r}(1)$ and target mean $m \in [\hat{\mu}(r), \mathbf{E}H_0^r]$. In particular $K(\mu) \neq u_1(r)$ for all μ.

Let $\widetilde{\mu}$ be the largest type for which either $q_1^K(\mu) \neq 1$ or $q_1^K(\mu) \neq 0$. As $\mathbf{E}H_0^r < 1$ for $r < \hat{r}(1)$ it must be that $\widetilde{\mu} > \mu^K$.

If q_1^K is such that $K(\mu) \neq 0$ on $\mu \in [0, \mu^K)$ then $m \in (\hat{\mu}(r), \mathbf{E}H_0^r]$. Moreover, by monotonicity of K it must be that $\mu^* < \mu^K$. Where μ^* is the largest type for which $K(\mu) = 0$.

The allocation can be perturbed on an interval $[\widetilde{\mu} - \widetilde{\delta}, \widetilde{\mu})$ such that the new allocation is $(1,0)$ on $[\widetilde{\mu} - \widetilde{\delta}, \widetilde{\mu})$. Where $\widetilde{\delta} > 0$ is arbitrarily small. For small enough $\widetilde{\delta}$ there exists $\delta, \delta^* > 0$ such that $\mu^* + \delta^* + \delta < \mu^K$ and the following holds

$$\int_{\mu^*}^{\mu^*+\delta^*} \mu q_1^K(\mu)\, dF(\mu) = \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \mu(1 - q_1^K(\mu))\, dF(\mu)$$

and

$$\int_{\mu^*+\delta^*}^{\mu^*+\delta^*+\delta} (1-\mu) \min\left\{1, \alpha_r(\mu^*+\delta^*)\frac{K(\mu^*+\delta^*)}{2u_1(r)}\right\} dF(\mu) = \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} (1-\mu) q_0^K(\mu)\, dF(\mu)$$

Thus, the allocation can be further perturbed on $[\mu^*, \mu^* + \delta + \delta^*)$ by allocating types in $[\mu^*, \mu^* + \delta^*)$ to $(0,0)$ and types in $[\mu^* + \delta^*, \mu^* + \delta + \delta^*)$ to $\left(\frac{K(\mu)}{u_1(r)}, \min\left\{1, \alpha_r(\mu^*+\delta^*)\frac{K(\mu^*+\delta^*)}{2u_1(r)}\right\}\right)$.

By construction, this perturbation preserves Equation 45, Equation 29 and Equation 48. Moreover, the revenue of the resulting perturbed allocation is greater than the original allocation q_1^K. The improvement in revenue follows from noting that non-decreasing $\frac{1-F(\mu)}{\mu f(\mu)}$ and $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ implies the following inequalities respectively

$$\int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \mu \left[(u_1(r) - R_0) - \frac{1}{\mu}\frac{1-F(\mu)}{f(\mu)}u_1(\mu)\right] q_1^K(\mu)\, dF(\mu)$$
$$\geq \int_{\mu^*}^{\mu^*+\delta^*} \mu \left[(u_1(r) - R_0) - \frac{1}{\mu}\frac{1-F(\mu)}{f(\mu)}u_1(\mu)\right] (1 - q_1^K(\mu))\, dF(\mu)$$

$$\implies \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \left[\mu(u_1(r) - R_0) - \frac{1-F(\mu)}{f(\mu)}u_1(\mu)\right] q_1^K(\mu)\, dF(\mu)$$
$$\geq \int_{\mu^*}^{\mu^*+\delta^*} \left[\mu(u_1(r) - R_0) - \frac{1-F(\mu)}{f(\mu)}u_1(\mu)\right] (1 - q_1^K(\mu))\, dF(\mu)$$

and

$$\int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} (1-\mu) \left[(u_1(r) - R_0) + \frac{1-F(\mu)}{(1-\mu)f(\mu)}u_1(\mu)\right] q_0^K(\mu)\, dF(\mu)$$
$$\leq \int_{\mu^*+\delta^*}^{\mu^*+\delta^*+\delta} (1-\mu) \left[(u_1(r) - R_0) + \frac{1-F(\mu)}{(1-\mu)f(\mu)}u_1(\mu)\right] \min\left\{1, \alpha_r(\mu^*+\delta^*)\frac{K(\mu^*+\delta^*)}{2u_1(r)}\right\} dF(\mu)$$

$$\implies \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \left[(1-\mu)(u_1(r) - R_0) + \frac{1-F(\mu)}{f(\mu)}u_1(\mu)\right] q_0^K(\mu)\, dF(\mu)$$
$$\leq \int_{\mu^*+\delta^*}^{\mu^*+\delta^*+\delta} (1-\mu) \left[(u_1(r) - R_0) + \frac{1-F(\mu)}{f(\mu)}u_1(\mu)\right] \min\left\{1, \alpha_r(\mu^*+\delta^*)\frac{K(\mu^*+\delta^*)}{2u_1(r)}\right\} dF(\mu)$$

□

Proof of Lemma 8. By Lemma 7 we can restrict attention to allocations with $K(\mu) = 0$ on $\mu \leq \mu^K$. Consider q_1^K such that $K(\mu) \neq 0$ on $[\mu^K, \mu_0]$ and $\mu^K < \mu_0$. As $\mu^K < \mu_0$ we get that the q^K corresponds to the relaxed problem with $x > 0$ and hence a target mean $m < \mathbf{E}H_0^r$.

Let μ^* be the largest type for which $K(\mu) = 0$. By monotonicity of K we get that $\mu^* < \mu_0$. For small enough $\delta^* > 0$ we can perturb the allocation on $(\mu^*, \mu^* + \delta^*]$

into the allocation $(0,0)$. As types in $(\mu^*, \mu^* + \delta^*]$ are indifferent between participating or not, the perturbation does not affect the gross surplus and reduces the information rent by flattening the slope K on $(\mu^*, \mu^* + \delta^*]$. But the perturbation might violate Equation 28 and Equation 29 as the increase in mean from the perturbation might be above $\mathbf{E}H^r_{x'}$ where x' is the size of the point mass at $\hat{\mu}(r)$ of the perturbed allocation.

To restore these constraints, we further perturb the allocation by decreasing $q^K_0(\mu)$ by $\varepsilon > 0$ on $[\mu^* + \delta^*, \mu_0)$. The size of the mass point at $\hat{\mu}(r)$ jumps from x to 0 discontinuously as $\varepsilon > 0$. The mean from the two perturbations increases continuously in δ^* and ε. By combining the two perturbations Equation 28 and Equation 29 hold for $m' \in [m, \mathbf{E}H^r_0]$ and $x = 0$.

But the combination of these might violate the monotonicity of the slope. To address this, consider highest type $\widetilde{\mu}(\varepsilon) \geq \mu_0$ for which $K(\mu) \geq K(\mu_0) + u_0(r)\varepsilon$. By monotonicity of K, $\widetilde{\mu}(\varepsilon)$ is increasing in ε and equal to μ_0 at $\varepsilon = 0$. On $[\mu_0, \widetilde{\mu}(\varepsilon)]$ the allocation can be further perturbed by increasing $q^K_1(\mu)$ and/or decreasing $q^K_0(\mu)$ where these changes increase the slope for type μ from $K(\mu)$ to $K(\mu_0) + \varepsilon$. Note that this perturbation also increases the mean continuously in ε.

The combination of the three perturbations is feasible for the relaxed problem. The first perturbation does not affect the gross surplus. The second and the third perturbations increase the gross surplus. Moreover, ε and δ^* can be chosen such that the increase in the information rent from the second and third perturbation is offset by the decrease in information rent from the first perturbation. In particular, the following inequality holds

$$\begin{aligned}&\int_{\mu^*}^{\mu^*+\delta^*} \frac{1-F(\mu)}{f(\mu)} K(\mu) dF(\mu) \\ &\quad \geq \int_{\mu^*+\delta^*}^{\mu_0} u_0(r)\varepsilon \frac{1-F(\mu)}{f(\mu)} dF(\mu) + \int_{\mu_0}^{\widetilde{\mu}(\varepsilon)} \left(K(\mu_0) + \varepsilon - K(\mu)\right) \frac{1-F(\mu)}{f(\mu)}\, dF(\mu)\end{aligned}$$

The existence of this revenue-increasing and feasible perturbed allocation shows that q^K is not the solution to the relaxed problem, which establishes the lemma. □

Proof of Lemma 9. If K changes in value on $[\overline{\mu}, 1)$, by monotonicity of K there exists $\widetilde{\mu}$ in $[\overline{\mu}, 1)$ and $\delta^* > 0$ such that $K(\overline{\mu} + \delta^*) < K(\widetilde{\mu})$. In particular $K(\overline{\mu}) \leq K(\overline{\mu} + \delta^*) < u_1(r)$.

This implies that for small enough $\varepsilon > 0$ the allocation resulting from perturbing q^K on $[\overline{\mu},\ \overline{\mu}+\delta^*)$ by $+\left(\varepsilon,\ \varepsilon\frac{u_1(r)}{u_0(r)}\right)$ for small enough $\varepsilon > 0$ preserves the slope K, Equation 14 and Equation 29.

This perturbation, however, increases the mean, and the new allocation can thus violate Equation 28. To restore the target mean, the allocation can be further perturbed by increasing q_0^K slightly for all types μ in $[\widetilde{\mu}, 1)$. Let $\varepsilon' > 0$ be the amount of this increase, and satisfies the following condition

$$\varepsilon' \int_{\overline{\mu}}^{\widetilde{\mu}} dF(\mu) = \varepsilon \int_{\overline{\mu}}^{\overline{\mu}+\delta^*} \mu\frac{1-m}{m} - (1-\mu)\frac{u_1(r)}{u_0(r)}\, dF(\mu)$$

The right hand side is positive as $\overline{\mu} \geq \frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}$. The value of $\varepsilon' > 0$ is arbitrarily small by choice of δ^* and ε, and hence Equation 14 and Equation 29 hold. By construction Equation 28 also holds. The first statement of the claim follows from noting that the combination of the two perturbations increases the gross surplus and decreases the rents.

To establish that K equals $\lim_{\mu\uparrow\overline{\mu}} K(\mu)$ or 1, we can then use our usual argument of reducing the allocation q_1^K on $[\mu_0, \mu_0+\delta)$ and increasing q_1^K on an interval $[1-\widetilde{\delta}, 1]$ for some $\delta, \widetilde{\delta} > 0$ while preserving the mean constraint Equation 28. Like before, we can show that as $\frac{1-F(\mu)}{\mu f(\mu)}$ non-increasing, this perturbation is revenue improving as it reduces the information rent. ☐

Proof of Lemma 10. From Lemma 7 and Lemma 8 it suffices to consider q^K such that $K(\mu) = 0$ for all types in $[0, \mu_0)$ Let $\widetilde{\mu}$ be the smallest type for which $q_1^K(\mu) = 1$. If $K(\mu)$ is not constant and equal to $\alpha_r^{-1}(\mu_0)u_1(r) - u_0(r)$ in $[\mu_0, \mu_1)$ by Equation 14 we have $\frac{K(\mu_0)+u_0(r)}{u_1(r)} > \alpha_r^{-1}(\mu_0)$. Thus there exist arbitrarily small δ^*, $\widetilde{\delta} > 0$ such that $\mu_0 + \delta^* < \widetilde{\mu} - \widetilde{\delta}$ and

$$\int_{\mu_0}^{\mu_0+\delta^*} \mu\left(\frac{K(\mu)+u_0(r)}{u_1(r)} - \alpha_r^{-1}(\mu_0)\right) dF(\mu) = \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \mu(1-q_1^K(\mu))\, dF(\mu)$$

The allocation can be perturbed by decreasing $q_1^K(\mu)$ to $\alpha_r^{-1}(\mu_0)$ on $[\mu_0, \mu_0+\delta^*]$ and by increasing $q_1^K(\mu)$ to 1 on $[\widetilde{\mu}-\widetilde{\delta},\ \widetilde{\mu})$. This preserves Equation 28, Equation 14 and Equation 29. The claim follows from noting that the perturbation weakly increases revenue as

$$\int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \mu \left[(u_1(r) - R_0) - \frac{1}{\mu}\frac{1-F(\mu)}{f(\mu)}u_1(\mu)\right] \left(\frac{K(\mu)+u_0(r)}{u_1(r)} - \alpha_r^{-1}(\mu_0)\right)(\mu)\, dF(\mu)$$
$$\geq \int_{\mu^*}^{\mu^*+\delta^*} \mu \left[(u_1(r) - R_0) - \frac{1}{\mu}\frac{1-F(\mu)}{f(\mu)}u_1(\mu)\right] (1-q_1^K(\mu))\, dF(\mu)$$

$$\implies \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \left[\mu(u_1(r) - R_0) - \frac{1-F(\mu)}{f(\mu)}u_1(\mu)\right] \left(\frac{K(\mu)+u_0(r)}{u_1(r)} - \alpha_r^{-1}(\mu_0)\right) dF(\mu)$$
$$\geq \int_{\mu^*}^{\mu^*+\delta^*} \left[\mu(u_1(r) - R_0) - \frac{1-F(\mu)}{f(\mu)}u_1(\mu)\right] (1-q_1^K(\mu))\, dF(\mu)$$

□

Proof of Lemma 11. If $K(\mu) \in (K(\mu_1), K(\overline{\mu}))$ on $[\mu_1, \overline{\mu}]$ we can construct a profitable perturbation by decreasing $q_0^K(\mu)$ by $\frac{K(\overline{\mu})-K(\mu)}{u_0(r)}$ on $[\overline{\mu}-\widetilde{\delta}, \overline{\mu})$ and by increasing $q_0^K(\mu)$ by $\frac{K(\mu)-K(\mu_1)}{u_0(r)}$ on $[\mu_1, \mu_1+\delta^*)$. Where $\widetilde{\delta}, \delta^* > 0$ are such that $\mu_1 + \delta^* < \overline{\mu} - \widetilde{\delta}$ and satisfy the following

$$\int_{\mu_1}^{\mu_1+\delta^*} (1-\mu)\frac{K(\overline{\mu})-K(\mu)}{u_0(r)}\, dF(\mu) = \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} (1-\mu)\frac{K(\mu)-K(\mu_1)}{u_0(r)}\, dF(\mu)$$

This perturbation preserves incentive compatibility and obedience and yields a greater revenue as $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing. □

7 Uniform Prior–Theorem 4

Proof of Theorem 4: Let the borrower's net payoff from rate r be given by $\hat{u}_\theta(r) = u_\theta(r) - R_0$. For $F \sim \text{Unif}[0,1]$ the conditions in Proposition 3 hold . The rating agency's payoff from allocation q and rate r can be expressed as

$$\Pi(q) = \int_0^1 (2\mu - 1)\hat{u}_1(r)q_1(\mu)\, d\mu - 2\int_0^1 (1-\mu)\hat{u}_0(r)q_0(\mu)\, d\mu$$

Let Equation 45 hold for some m in $[\hat{\mu}(r), \mathbf{E}H_0^r]$. Substituting it in the above expression yields

$$\Pi(q) = \int_0^1 \left[(2\mu-1)\hat{u}_1(r) - 2\mu\frac{1-m}{m}\hat{u}_0(r)\right] q_1(\mu)\, d\mu$$

As $u_0(r) - R_0 \leq 0$ for all $r \geq \hat{r}(\mu)$ we get

$$\Pi(q) \leq \int_0^1 (2\mu - 1)\hat{u}_1(r) q_1(\mu)\, d\mu \leq \int_{\frac{1}{2}}^1 (2\mu - 1)\hat{u}(\hat{r}(0))\, d\mu$$

The upper bound above is achieved by the following allocation

$$q^*(\mu) := \begin{cases} (0,0) & \mu < \frac{1}{2} \\ \\ (1,1) & \mu \geq \frac{1}{2} \end{cases}$$

The corresponding $\text{marg}_{\Delta(\Theta)}\, \beta$ is

$$\text{marg}_{\Delta(\theta)}\, \beta(\mu \mid r = \hat{r}(0)\ ) = \begin{cases} 0 & \mu < 0.5 \\ \\ 2\mu - 1 & \mu \geq 0.5 \end{cases}$$

Consider the distribution $H_{0.25}^{\hat{r}(0)} = \mathit{Bernoulli}(0.75)$. From Figure 2 we conclude that $\text{marg}_{\Delta(\theta)}\, \beta(\mu \mid r = \hat{r}(0)\ )$ crosses $H_{0.25}^{\hat{r}(0)}$ once and from below, thus $H_{0.25}^{\hat{r}(0)} \succeq_{\text{mps}} \text{marg}_{\Delta(\theta)}\, \beta(\mu \mid r = \hat{r}(0)\ )$.

Finally, note that under Equation 31 the distribution $H_{0.25}^{\hat{r}(0)}$ satisfies the obedience constraint Equation 21 for $r = \hat{r}(0)$. □