Incentive-Compatible Information Design*

Hershdeep Chopra † Jeffrey C. Ely ‡

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Abstract

We study the design of mechanisms by an intermediary that generates information for a sender to persuade a receiver about an unknown attribute of the sender. The sender is initially privately, but imperfectly, informed about her attribute, and the receiver takes an action based on posterior beliefs about the sender's attribute and the sender's belief about the attribute. The mechanism generates information for the sender and also controls its disclosure to the receiver. The design of the optimal mechanism needs to screen the privately informed sender and thus confronts incentive-compatibility constraints. The mechanism also deals with obedience constraints, as the intermediary must generate just enough information to persuade the receiver. We characterize incentive-compatible mechanisms for a wide class of problems when the sender contracts with the intermediary. We use this characterization to study profit-maximizing mechanisms in three applications: the design of college-admissions tests, the optimal use of consumer data on a digital market platform, and the optimal design of credit rating schemes.

1 Introduction

In recent decades, economic theory has developed sophisticated tools to study the extent to which information provision can affect the behavior and interaction of economic agents. Yet much of the attention has been focused on scenarios where

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†Department of Economics, Northwestern University. hershdeepchopra2026@u.northwestern.edu.

‡Department of Economics, Northwestern University. jeff@jeffely.com. Support from the National Science Foundation is gratefully, albeit nervously, acknowledged.

the provision of information is through commitment, an omniscient (information) designer, or other exogenous considerations. This precludes environments where agents have to acquire information, often at a cost and from an intermediary. We study a general class of problems in which an intermediary sells an *information policy* to a privately but imperfectly informed sender who will interact *ex post* with a receiver. The allocation mechanism must be incentive-compatible, which restricts what information is available to the sender. The design of information subject to incentive compatibility of the allocation mechanism is the pervasive theme of this paper. We specialize our theoretical framework to study profit-maximizing information policies in three instances: quality signaling with tests, monopoly pricing on online platforms, and risk mitigation through credit rating.

In our first application, we consider a firm that designs a college admissions test and sets a price to students for taking the test. The design of the test can reveal more or less information about the student's ability. Variations in the informativeness of the test affect how the college uses the test in admissions decisions, and this affects the student's willingness to pay for the test. In addition to these tradeoffs, the students have prior private (but imperfect) information about their ability and this translates into private information about how they will perform on the test and thus how much they are willing to pay for the test. The design of the test must therefore grapple with screening frictions. These screening frictions are novel owing to the particular features of a test as a screening instrument.

Next we consider the design of a data policy by a digital market platform. The platform can use the data it collects about users to estimate consumers' values for new products. This information is useful both to the consumers on the platform (to improve their purchase decisions) and to sellers on the platform (to target buyers and set prices). The platform can, for example, offer data management policies to consumers which help consumers make informed purchases and which also incentivize firms to set prices that maintain high consumer surplus. Consumers are willing to pay for such policies, and this can be a source of revenue for the platform. At the same time the consumers have their own private information about willingness to pay. The value of any data-management policy will be different for consumers with different private beliefs and so the platform faces screening frictions similar to those of the test-design firm. Data management policies are effective information design policies and these screening frictions are also disciplined by similar constraints.

In our third application, we look at the design of credit rating schemes in a loan market. The rating agency evaluates the riskiness of an investment project. In the presence of a risk-free outside option, less risky borrowers might forgo investing when facing high enough interest rates. Thus, credit ratings affect whether the borrower seeks to invest in the project or exercise her outside option. Credit ratings also inform the lender of the borrower's risk profile and can be used to increase the borrower's surplus from the loan. Borrowers are willing to pay for credit ratings, and the rating agency can design them to maximize its revenue. Like the previous examples, the rating agency faces screening frictions as the borrower's private information about the riskiness of the project affects her willingness to pay for the credit rating scheme.

These three applications are representative of the larger class of problems involving information acquisition by a privately informed sender from a profit-maximizing monopolistic intermediary and information disclosure to influence the receiver's decisions. The intermediary can design a policy that enhances the information of the sender while at the same time selectively disclosing information to the receiver to influence her ex post action. We combine tools of mechanism design and information design to characterize incentive-compatible information policies coupled with obedient decision rules for the receiver. We use these tools to solve for optimal mechanisms in the three examples described above.

The design of an optimal mechanism is inherently a multi-dimensional screening problem (for example, how finely to discriminate between students in the upper tail versus students in the lower tail). We provide the necessary conditions for a mechanism to be incentive compatible in terms of monotonicity of a functional of the information policy (equivalent to convexity of the sender's indirect utility) prescribed by the mechanism. These conditions are also sufficient when the sender's payoff is linear in her type.

In our college admission application, the student's utility is independent of her ability, as she only values being admitted to the college regardless of her ability. The school (receiver) chooses to admit or reject the student based on its expectation of the student's ability. We use our characterization of incentive compatibility to enumerate the extreme points of the set of incentive-compatible, obedient, and individually rational information policies. We then use these extreme points to give a qualitative description of the optimal mechanism. We characterize the structure of the optimal mechanism under two conditions on the receiver's prior, F, about the student's ability. Under the first condition, the optimal mechanism provides more informative (harder) tests to higher ability students and leaves rents only to the higher types. Whereas under the second condition, the optimal mechanism spreads out the informativeness (difficulty) of the tests more evenly among different types of students and leaves rents to all unexcluded types.

In our other applications, the sender's utility is no longer linear in her type, as the sender might prefer to exercise her outside option over accepting the action recommended by the mechanism to the receiver during the ex-post interaction of the sender and the receiver. For example, a consumer who learns that the product's value is low prefers not buying the item at a high enough price. Similarly, a borrower with low risk tolerance prefers her safe option over borrowing at a high interest rate. In both these examples, we restore the linearity of the sender's payoff in her type by introducing an additional set of constraints to the mechanism design problem. These constraints enforce the sender's participation by ruling out the sender exercising her outside option in the ex-post interaction with the receiver. Introducing these additional constraints allows us to use our characterization of incentive compatibility.

Unlike our college admission application, in the other two applications, the receiver's action depends on the whole distribution of the sender's type (receiver's second-order beliefs), which presents additional challenges in dealing with obedience constraints. We introduce a new family of iso-elastic distributions, such that any obedient information policy satisfying the sender's participation constraint can be transformed via information disclosure into a distribution in this family. We then relax the problem by replacing the obedience constraints with two necessary conditions for the receiver's second-order beliefs to be a mean-preserving contraction of some distribution in the iso-elastic family.

We solve for the optimal mechanism of the relaxed problem by first fixing the slope of the sender's indirect utility and deriving the optimal information policy corresponding to that slope. Then we optimize over the slope to recover the optimal mechanism for the relaxed problem. Our approach is valuable as it reduces the difficulty of calculating an optimal mechanism to verifying a single crossing condition between the second-order beliefs of the receiver resulting from the relaxed optimal mechanism and a distribution in the iso-elastic family. We demonstrate this by verifying the said single crossing when the receiver has a uniform prior.

2 Model

Setup A sender has attribute $\theta \in \{0, 1\}$, unknown to all. The sender is privately but imperfectly informed of θ as represented by her prior belief μ. Only the sender knows her prior μ, and we will refer to μ as the type of the sender.

There is a receiver who is initially uninformed about the sender's attribute θ and her type μ, knowing only that the latter is distributed according to a prior distribution function F which is continuous and has density f.

An intermediary will design an information policy which will generate new information about θ. The intermediary can reveal some of this information to the sender, allowing the sender to update her belief to a posterior ν. The intermediary can also reveal information to the receiver. This will allow the receiver to form a *second-order* belief, i.e. an element of $\Delta(\Theta \times \Delta\Theta)$, about both the attribute θ and the posterior belief ν of the sender.

After observing the information provided by the intermediary, the receiver will take an action $a \in A$ and obtain a payoff $\pi_R(a, \beta)$. The sender will earn payoff $\pi_S(a, \nu)$.

Examples In our first application the sender is a student and the receiver is a college. The intermediary sells a test to the student which will reveal information to the college to be used in admissions decision. Here θ is the student's ability, high ($\theta = 1$) or low ($\theta = 0$). The college chooses from the pair of actions $A = \{\texttt{admit}, \texttt{reject}\}$ and wants to admit students of high ability. Specifically the *ex-post* payoff from admitting the student depends only on θ and is given by the following matrix where $p \in (0, 1)$.

	$\theta = 0$	$\theta = 1$
`admit`	-1	$\frac{1-p}{p}$
`reject`	0	0

Given a belief β the college makes its admissions decision to maximize its expected payoff. In particular the student is admitted if and only if the college beleives with at least probability p that the student has high ability. The student earns a payoff of 1 from being admitted and 0 from being rejected, independent of θ.

To avoid trivialities we assume $\mathbf{E}_F\theta < p$ so that some information is necessary to induce the college to admit the student.

Our second application is to digital market platforms. Here the sender is a buyer whose willingness to pay for a product is θ. The receiver is the seller of the product; he offers a price p which the buyer will either accept or reject. Rejection leads to a payoff of zero for both parties and acceptance leads to payoff $\theta - p$ for the buyer and p for the seller. Thus, when the buyer has posterior ν, she will accept any price $p \leq \nu$ and her total payoff will be $\max\{\nu - p, 0\}$. Given belief β, the seller's payoff from offering price p is given by

$$p \cdot \text{Prob}(\nu \geq p).$$

Note that the seller's payoff depends on his second-order belief: his belief about the buyer's belief.

In our final application, we consider a credit rating agency. The sender is a borrower seeking to take out a loan to fund a project, the amount of the loan is normalized to 1. The receiver is a lender who offers to fund the project at an interest rate r. In state θ, the distribution of returns from funding the project is G_θ. Where G_1 is a mean preserving spread of G_0. The expected return of the projects in either state is $\overline{R} > 1$, but the riskiness of the project is increasing in θ. The borrower chooses between exercising an outside option $R_0 < 1$ or borrowing at the interest rate r with limited liability. When the realized return from the project is greater than $1 + r$, the payoff of the lender is $1 + r$, and the payoff of the borrower is the return minus $1 + r$. If the realized return of the project is less than $1 + r$, then the borrower defaults. In this case, the lender recovers the realized returns of the project and the borrower has zero payoff.

Given a rate r, the borrower's payoff is increasing in his posterior ν (riskiness). Let $\hat{\mu}(r)$ be the posterior that makes the borrower indifferent between borrowing at rate r and exercising her outside option. Thus, a borrower with posterior ν accepts a loan at rate r whenever $\nu \geq \hat{\mu}(r)$ and has a total payoff

$$\mathbf{E}_\nu \left[\int \max\{R - (1 + r),\ 0\} dG_\theta(R) \right]$$

Given belief β, the lender's payoff from offering a interest rate r is given by

$$\mathbf{E}_\beta \left[\int \min\{R - 1,\ r\} dG_\theta(R) \,\middle|\, \nu \geq \hat{\mu}(r) \right] \cdot \text{Prob}(\nu \geq \hat{\mu}(r))$$

Information Policies A *test* is defined by two non-empty sets of messages ("test results") $\mathcal{M}_S$ and $\mathcal{M}_R$ and for each θ a distribution $\rho_\theta \in \Delta(\mathcal{M}_S \times \mathcal{M}_R)$. When the sender submits to a test, the results are drawn from ρ_θ conditional on the sender's true attribute. The result in $\mathcal{M}_S$ is privately disclosed to the sender and the result in $\mathcal{M}_R$ is privately disclosed to the receiver. An information policy, or a *mechanism* is defined by a non-empty set of messages $\mathcal{M}_\Sigma$ and a rule that specifies as a function of $\mathcal{M}_\Sigma$ both a payment made by the receiver and a test. The receiver selects a message from $\mathcal{M}_\Sigma$ and the resulting test is carried out. Importantly, we restrict payments to be *ex ante*, i.e. before the realization of the test.

In a *direct* mechanism $\mathcal{M}_\Sigma = \Theta$, $\mathcal{M}_R = A$ and $\mathcal{M}_S = \Delta\theta$. The sender reports her type $\mu \in \Theta$ and in response the intermediary charges the sender the fee $\phi(\mu)$, uses $\rho_\theta(\mu)$ to recommend an action $a \in A$ to the receiver and privately informs the sender of her posterior $\nu \in \Delta\theta$. Thus, $\rho_\theta(\mu) \in \Delta(A \times \Delta\theta)$. When the intermediary uses a direct mechanism and the realized recommendation is a, the receiver formulates the conditional belief $\beta(\cdot \mid a) \in \Delta(\theta \times \Delta\theta)$.

A direct mechanism is *obedient* if the recommended action is always the one that maximizes the receiver's payoff conditional on hearing the recommendation, i.e. with probability 1

$$a \in \operatorname{argmax}_A \pi_R(\cdot, \beta(\cdot \mid a)).$$

Let $\tilde{\nu}(\mu) = \mu \operatorname{marg}_{\Delta\theta} \rho_1(\mu) + (1-\mu) \operatorname{marg}_{\Delta\theta} \rho_0(\mu)$ be the total probability distribution of the posterior induced for type μ. A direct mechanism is *bayes-plausible* if for every μ,

$$\mathbf{E}\tilde{\nu}(\mu) = \mu.$$

Recall that Kamenica and Gentzkow (2011) bayes-plausibility implies that for a sender with prior μ, the conditional probability of $\theta = 1$ conditional on every suggested posterior ν is ν itself. On the other hand, should μ report some other type $\mu' \neq \mu$ to the mechanism and receive the posterior ν suggested for μ', she would use that suggestion to update her true prior μ and thereby obtain a possibly different posterior, call it $\delta_{\mu,\mu'}(\nu)$. By continuity of Bayesian updating, $\delta_{\mu,\mu'}(\nu)$ is continuous in μ for any given μ' and ν.

A direct mechanism yields *gross utility* for type μ equal to $v(\mu) = \mathbf{E}_{\rho(\mu)}\pi_S(a, \nu)$. We define the *indirect utility* function of a direct mechanism by

$$U(\mu) = v(\mu) - \phi(\mu)$$

When μ misreports $\mu' \neq \mu$ she instead earns gross utility

$$v(\mu' \mid \mu) = \mu \mathbf{E}_{\rho_1(\mu')}\pi_S(a, \delta_{\mu,\mu'}(\nu)) + (1-\mu)\mathbf{E}_{\rho_0(\mu')}\pi_S(a, \delta_{\mu,\mu'}(\nu))$$

A direct mechanism is *incentive compatible* if for every μ, μ'

$$U(\mu) \geq v(\mu' \mid \mu) - \phi(\mu').$$

A direct mechanism is *individually rational* if $U(\mu) \geq 0$ for all μ.

The goal of the intermediary is to maximize expected revenue. By the revelation principle we can restrict attention to obedient, bayes-plausible and incentive-compatible direct mechanisms (ρ, ϕ) designed to maximize

$$\Pi = \int_0^1 \phi(\mu) dF(\mu)$$

3 Characterization of Incentive Compatibility

The gross deviation utility $v(\mu' \mid \mu)$ is non-linear which complicates the usual envelope representation of incentive compatible mechanisms. However we can make use of the following observation to obtain a condition that is necessary in general and sufficient in some special cases.

Let (ρ, ϕ) be an incentive-compatible mechanism. Suppose for every μ' there exist numbers $q_0(\mu')$ and $q_1(\mu')$ such that for every μ

$$v(\mu' \mid \mu) \geq \mu q_1(\mu') + (1 - \mu) q_0(\mu') \tag{1}$$

with equality when $\mu = \mu'$. Then the linear function on the right-hand size is a support function for the indirect utility function:

$$\begin{aligned} U(\mu) &\geq v(\mu' \mid \mu) - \phi(\mu') \\ &\geq \mu q_1(\mu') + (1 - \mu) q_0(\mu') - \phi(\mu') \\ &= (\mu - \mu') q_1(\mu') + \left[(1 - \mu) - (1 - \mu')\right] q_0(\mu') + \mu' q_1(\mu') + (1 - \mu') q_0(\mu') - \phi(\mu') \\ &= U(\mu') + (\mu - \mu') \left[q_1(\mu') - q_0(\mu')\right] \end{aligned}$$

This implies that $U(\cdot)$ is convex and hence absolutely continuous. It is differentiable at almost every μ with slope $q_1(\mu) - q_0(\mu)$. The indirect utility of type μ can be expressed as

$$U(\mu) = U(0) + \int_0^\mu \left[q_1(\mu') - q_0(\mu')\right] d\mu'$$

Turning things around, suppose that a mechanism (ρ, ϕ) yields a convex indirect utility function U and the inequality in Equation 1 holds with equality for all μ. If the integral representation above holds then the mechanism is incentive compatible because by convexity.

$$U(\mu) \geq U(\mu') + (\mu - \mu') \left[q_1(\mu') - q_0(\mu')\right] = v(\mu' \mid \mu) - \phi(\mu').$$

4 College Admissions

In the collge admissions application a direct mechanism recommends `admit` or `reject` to the college and the obedience constraint is

$$\text{Prob}(\theta = 1 \mid \alpha = \texttt{admit}) \geq p$$

Let $q_\theta(\mu) = \text{marg}_A\, \rho_\theta(\mu)\, [\texttt{admit}]$ be the probability that a student with type μ and ability θ is recommended for admission. Then

$$v(\mu \mid \mu') = \mu q_1(\mu') + (1-\mu) q_0(\mu')$$

for all μ and μ', and therefore the monotonicity condition, namely for all $\mu \geq \mu'$

$$q_1(\mu) - q_0(\mu) \geq q_1(\mu') - q_0(\mu')$$

together with the envelope formula are necessary and sufficient condition for incentive compatibility.

For any incentive compatible mechanism there must exist a type μ^0 whose participation constraint binds, i.e. $U(\mu^0) = 0$. If not, fees could be increased by a constant for all types, raising revenue without altering incentive constraints.

Then we may write

$$U(\mu) = \begin{cases} -\int_\mu^{\mu^0} [q_1(\mu') - q_0(\mu')]\, d\mu', & \text{if } \mu \leq \mu^0 \\ \int_\mu^{\mu^0} [q_1(\mu') - q_0(\mu')]\, d\mu', & \text{if } \mu \geq \mu^0 \end{cases}$$

Since U is convex and weakly positive, the slope must be non-positive below μ^0 and non-negative above. In particular $q_1(\mu) \leq q_0(\mu)$ for all $\mu \leq \mu^0$.

We may express the intermediary's expected revenue as expected gross utility minus expected indirect utility.

$$\Pi = \int_0^{\mu^0} \left\{ v(\mu) + \int_\mu^{\mu^0} \left[q_1(\mu') - q_0(\mu')\right] d\mu' \right\} f(\mu) d\mu \\ + \int_{\mu^0}^1 \left\{ v(\mu) - \int_\mu^{\mu^0} \left[q_1(\mu') - q_0(\mu')\right] d\mu' \right\} f(\mu) d\mu$$

We can see immediately that profit is increasing in $q_1(\mu)$ for all $\mu \leq \mu^0$. Moreover, increasing $q_1(\mu)$ adds slack to the obedience constraint and therefore the constraint $q_1(\mu) \leq q_0(\mu)$ must bind and we have $q_1(\mu) = q_0(\mu)$ for all $\mu \leq \mu^0$.

It follows that $q_1(\mu) \geq q_0(\mu)$ for all $\mu \in [0, 1]$ and in fact we may take $\mu^0 = 0$, that is $U(0) = 0$.

When we change the order of integration and simplify we obtain the following expression for expected *virtual surplus*

$$\Pi = \mathbf{E}\left[q_1(\mu)\left(\mu - \frac{1 - F(\mu)}{f(\mu)}\right)\right] + \mathbf{E}\left[q_0(\mu)\left(1 - \mu + \frac{1 - F(\mu)}{f(\mu)}\right)\right] \tag{2}$$

The problem reduces to maximizing expected virtual surplus subject to monotonicity, obedience, and $q_1(\mu) \geq q_0(\mu)$.

Notice that the sum of the coefficients on $q_1(\mu)$ and $q_0(\mu)$ equals 1. If we ignore the obedience constraint, it would be optimal therefore to set $q_1(\mu) = q_0(\mu) = 1$ for all μ. This is the mechanism that admits the student with probability 1 (and extracts all of the surplus), but this mechanism violates obedience because $\mathbf{E}\mu < p$.

It follows that the obedience constraint is binding at the optimum and the problem becomes one of restoring obedience at minimum cost in terms of foregone virtual surplus, and subject to monotonicity and $q_1(\mu) \geq q_0(\mu)$. One candidate mechanism is a threshold mechanism in which $q_1(\mu) = q_0(\mu)$ for all μ and these admission probabilities jump from 0 to 1 at the interior point $\tilde{\mu}$ defined by

$$\mathbf{E}(\mu \mid \mu \geq \tilde{\mu}) = p$$

This mechanism is feasible because it is monotonic, and satisfies both obedience and $q_1(\mu) \geq q_0(\mu)$. Moreover its indirect utility has a constant slope of zero so the mechanism extracts all of the surplus it generates.

Nevertheless this mechanism is typically not optimal. Instead a mechanism which selectively rejects some low-ability students (by setting $q_1(\mu) > q_0(\mu)$ for some types), can generate larger revenues despite yielding positive rents to the student.

Proposition 1. *The optimal revenue is achieved by allocation of the following form, for some values* $0 \leq \mu_0 \leq \overline{\mu} \leq 1$.

1. *If* $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ *is non-increasing then optimal allocation* (q_1, q_0) *is of the form*

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu \in [0, \mu_0) \\ (1,1) & \mu \in [\mu_0, \overline{\mu}) \\ (1,0) & \mu \in [\overline{\mu}, 1] \end{cases}$$

   Where $\int_{\mu_0}^{1} \mu dF(\mu) = \frac{p}{1-p} \int_{\overline{\mu}}^{1} (1-\mu) dF(\mu)$.

2. *If* $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ *is non-decreasing then optimal allocation* (q_1, q_0) *is of the form*

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu \in [0, \mu_0) \\ (1-\alpha, 0) & \mu \in [\mu_0, \overline{\mu}) \\ (1, \alpha) & \mu \in [\overline{\mu}, 1] \end{cases}$$

   Where $(1-\alpha) \int_{\mu_0}^{\overline{\mu}} \mu dF(\mu) + \int_{\overline{\mu}}^{1} \mu dF(\mu) = \alpha \frac{p}{1-p} \int_{\overline{\mu}}^{1} (1-\mu) dF(\mu)$.

The optimal mechanism obtained in item 1 allocates informative tests to the higher types, and lower types are offered uninformative tests. Students with lower prior expected ability take easier tests (less informative) and contribute to a greater share of the intermediate's revenue. Students with higher prior expected ability take difficult tests (more informative) and earn greater information rent. In contrast, the optimal mechanism obtained in item 2 allocates informative tests more evenly among different applicant types. In fact, there is a reversal in the allocation of perfectly revealing tests. For item 2, types in the middle are admitted only if the student is high-ability, and higher types are admitted even if they are low-ability. Whereas in item 1, types in the middle are always admitted and higher types are admitted only if they are high-ability. For a welfare perspective, all types that are not excluded by the intermediary earn information rents in item 2, whereas the intermediary extracts all surplus from types that are allocated uninformative tests in item 1.

5 Digital Platform

Our next application is to digital marketplace platforms. A platform has access to detailed data which enables it to precisely estimate a buyer's willingness to pay for products. The platform can provide this information to the buyer but moreover to the seller. The former guides the buyer's purchases while the latter disciplines the prices set by sellers. The platform sells access to the buyer and by controlling the information available to sellers the platform also controls the consumer surplus provided to buyers and thus increases the value of membership on the platform.

Formally the buyer plays the role of the sender, with willingness to pay θ, and private information μ. The seller is the receiver, setting a price $p \in A = \mathbf{R}$. A sale at price p yields $\theta - p$ for the buyer and p for the seller. The payoff to both from no sale (including when the buyer is excluded from the platform) is zero.

A mechanism for the platform specifies whether the buyer will be admitted to the platform and if so provides information to the buyer leading to the posterior ν and recommends the price p to the seller. The seller infers the distribution of posteriors ν for buyers on the platform and adopts the recommended price p if and only if

$$p \cdot \text{Prob}(\nu \geq p \mid p) \geq p' \cdot \text{Prob}(\nu \geq p' \mid p) \tag{3}$$

The above system of inequalities constitutes the obedience constraint.

Participation Let $q_\theta(\mu)$ denote the probability that type μ joins the platform when having willingness to pay θ, and let

$$\overline{q}(\mu) = \mu q_1(\mu) + (1 - \mu) q_0(\mu)$$

be the total probability that μ joins the platform. The buyer earns non-negative payoff only when joining the platform. The participation constraint therefore requires that the expected payoff for every type μ conditional on joining the platform and paying price p is non-negative:

$$\mu \cdot q_1(\mu) \geq p\overline{q}(\mu) \tag{4}$$

Incentive Compatibility The following observation enables us to use our general representation of incentive compatibility. Since the buyer's payoff is zero when excluded from the platform this payoff is equivalent to the payoff from purchasing the good at price equal to her willingness to pay θ. Thus we may without loss represent any mechanism as one in which sale occurs with probability 1 at a random price from the set $\{0, p, 1\}$.

In particular the expected price paid by type μ having willingness to pay θ is

$$\bar{p}_1(\mu) = p \cdot q_1(\mu) + 1 \cdot (1 - q_1(\mu)) \tag{5}$$

for $\theta = 1$ and

$$\bar{p}_0(\mu) = p \cdot q_0(\mu) + 0 \cdot (1 - q_0(\mu)) \tag{6}$$

for $\theta = 0$. Since the total (both on and off the platform) expected value of the buyer's posterior is equal to her type μ, this yields gross payoff

$$v(\mu) = \mathbf{E}\,(\nu - p) = \mu - (1-\mu)\bar{p}_0(\mu) - \mu\bar{p}_1(\mu)$$

where $\bar{p}_\theta(\mu)$ is the expected value of the random price offered to type μ, conditional on having true willingness to pay θ. The deviation payoff is

$$v(\mu' \mid \mu) = \mu' - \mu'\bar{p}_1(\mu) - (1-\mu')\bar{p}_0(\mu) = \mu'(1 - \bar{p}_1(\mu)) - (1-\mu')\bar{p}_0(\mu)$$

and therefore the monotonicity condition requires that $1 - \bar{p}_1(\mu) + \bar{p}_0(\mu)$ is weakly increasing. Or using the expressions in Equation 5 and Equation 6,

$$K(\mu) = (1-p)q_1(\mu) + pq_0(\mu) \quad \text{is weakly increasing in } \mu. \tag{7}$$

This condition together with the envelope formula

$$U(\mu) = \int_0^\mu \left[(1-p)q_1(\mu') + pq_0(\mu')\right] d\mu' + U(0) \tag{8}$$

are necessary and sufficient for incentive compatibility.[1]

Relaxed Obedience Constraint In the previous examples, the receiver's conditional beliefs about θ were sufficient to characterize the obedience constraint. Here by contrast, obedience for the seller depends on the full conditional distribution of the buyer's interim beliefs. To facilitate the analysis of obedience we will make use of results in a companion paper Chopra and Ely (2025) which in turn builds on the results of Roesler and Szentes (2017) to characterize obedient value distributions when the buyer has some initial private information.

[1] With this formulation we are empowering the platform to enforce sale when the seller accepts the recommended price p. Thus, the mechanism needn't guard against "double deviations" in which a buyer of type μ misreports as type μ' and then selectively rejects p for some posteriors. Nevertheless, we show below that the optimal mechanism the platform never enforces undesirable purchases on path. Moreover, in Section B.7 we discuss the issue of double deviations further and show that allowing randomized prices deters all double deviations in any incentive-compatible and individually-rational mechanism.

Consider the following family of cumulative distribution functions parameterized by a target price p and a mass x.

$$H_x^p(s) = \begin{cases} 0 & s < p \\ x & s = p \\ 1 - \frac{p}{s} & 1 > s \geq \frac{p}{1-x} \\ 1 & s = 1 \end{cases}$$

Note that when $x = 0$ the distribution H_x^p is the unit-elastic demand studied by Roesler and Szentes (2017) and makes the seller indifferent between all prices in the interval $[p, 1]$. In other words H_0^p satisfies the obedience constraint Equation 3 with equality for all $p' \geq p$. As shown by Roesler and Szentes (2017) this is the way to maximize consumer surplus of a buyer with no initial private information.

In our setting the platform is a profit-maximizer screening a buyer with private information. The seller is therefore willing to tradeoff consumer surplus (i.e. efficiency) in exchange for rent extraction. Moreover given the buyer's initial private information there typically does not exist an information policy that can generate an iso-elastic demand. Instead we will show that obedience can more generally be satisfied by targeting information to the buyer in such a way as to generate a distribution of posteriors of the form H_x^p for some $x \geq 0$. The resulting value distribution makes the seller indifferent between all prices in $\{p\} \cup [p/(1-x), 1]$.[2]

A distribution of buyer values can be transformed into one that satisfies obedience if and only if it is a mean-preserving contraction of some H_x^p. We can therefore replace the obedience constraint with the condition that $\text{marg}_{\Delta\theta}\, \beta(\cdot \mid p)$ to be a mean-preserving contraction of H_x^p for some x. We will instead impose a relaxed constraint consisting of two necessary conditions. The first condition is that the mean valuation in $\text{marg}_{\Delta\theta}\, \beta(\cdot \mid p)$ equals the mean of H_x^p.

$$\mathbf{E}\, \text{marg}_{\Delta\theta}\, \beta(\cdot \mid p) = \mathbf{E} H_x^p. \tag{9}$$

When this condition holds and additionally the CDF of $\text{marg}_{\Delta\theta}\, \beta(\cdot \mid p)$ crosses H_x^p once and from below then the former is a mean-preserving contraction of the latter. A necessary condition for this single crossing is

$$\text{marg}_{\Delta\theta}\, \beta(p \mid p) \leq x, \tag{10}$$

[2] As it turns out however we identify a condition on the ex ante distribution F under which the optimal mechanism is indeed a unit-elastic demand with mass $x = 0$.

that is the CDF of $\text{marg}_{\Delta\theta}\, \beta(\cdot \mid p)$ is below H^p_x at the point p.

Note for future reference that $\mathbf{E}H^p_x$ is strictly decreasing in x.

Relaxed Problem When we express the platform's profit as the difference between expected gross payoff and expected buyer utility, represent expected utility by the envelope formula in with $U(0) = 0$[3] and then proceed with the standard manipulations we obtain the following objective function for the platform.

$$\Pi = \int_0^1 (1-p)\left(\mu - \frac{1-F(\mu)}{f(\mu)}\right) q_1(\mu) dF(\mu) - \int_0^1 p\left(1 - \mu + \frac{1-F(\mu)}{f(\mu)}\right) q_0(\mu) dF(\mu) \quad (11)$$

Our relaxed problem is to choose $q_\theta(\cdot)$ to maximize this profit function subject to monotonicity (Equation 7), participation (Equation 4), and relaxed obedience (Equation 9 and Equation 10).

Proposition 2. *If $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing, then for any p one of the following $q(\mu) = (q_1(\mu), q_0(\mu))$ is a solution to the relaxed problem*

$$q(\mu) = \begin{cases} (0,0) & \text{if} \quad \mu \le \mu_0 \\ \left(\frac{p}{1-p}\frac{1-\mu_0}{\mu_0}, 1\right) & \text{if} \quad \mu \in [\mu_0, \mu') \\ (1,1) & \text{otherwise} \end{cases} \quad \text{or} \quad q(\mu) = \begin{cases} (0,0) & \text{if} \quad \mu \le \mu_0 \\ (1,0) & \text{if} \quad \mu \in [\mu_0, \mu') \\ (1,1) & \text{otherwise} \end{cases}$$

for some thresholds $0 \le \mu_0 \le \mu' \le 1$

Notice that the likelihood ratio $q_1(\mu)/q_0(\mu)$ is greater than or equal to 1 for all types μ except those belonging to the second interval $[\mu_0, \mu')$. In that second interval the likelihood ratio is weakly increasing. It follows that there is at most one type such that Equation 4 holds with equality. More precisely, the distribution of values induced by the relaxed solution has no mass at p, i.e. $x = 0$. By Lemma 5 induced value distribution has the same mean as H^p_0, the iso-elastic demand implementing price p.

When we then ask whether the relaxed solution can be transformed into a fully obedient mechanism and therefore a solution to the original problem, we need

[3] From Equation 8 we see that $U(\mu) \ge U(0)$ for all μ and therefore if $U(0) > 0$ we could increase payments by the constant $U(0)$ without violating any constraints.

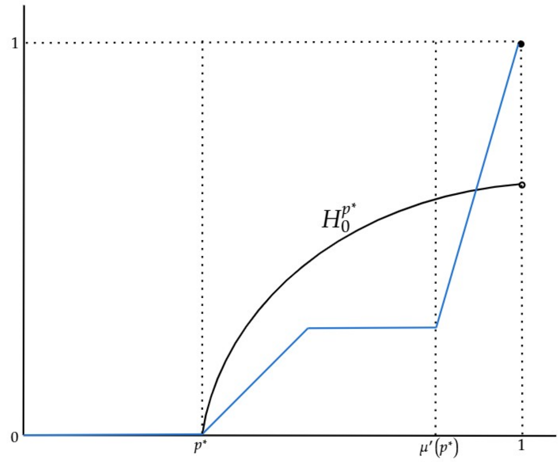

Figure 1: For uniform distribution, the optimal price $p^*(\approx 0.417)$ solves the relaxed problem and generates the distribution of buyer valuation, conditional on being recommended price p^*, represented above in blue. This distribution crosses $H_0^{p^*}$ once and from below and hence is a mean preserving contraction of $H_0^{p^*}$.

only check whether this value distribution crosses H_0^p once and from below. We show in the appendix (Section B.6) that this is true in particular for the uniform distribution on $[0, 1]$. The structure of the optimal mechanism for this case is represented in Figure 1.

Theorem 1. *If F is uniform on $[0, 1]$, then the solution to the relaxed problem in Proposition 2 with revenue-maximizing price p^* is also a solution to the original problem.*

6 Credit Rating

In this section, we present an application of our framework to credit rating in a loan market. Like our digital platform example (Section 5), the credit rating agency can provide information to both the borrower and the lender. The rating agency sells access to credit to the borrowers and informs the borrower about the riskiness of the project. The rating agency also selectively discloses this information to shape the lender's belief about the risk profile of the borrower conditional on being given access to credit. The rating agency can use its information policy to increase the borrower's surplus from the loan and thus increase the borrower's value from the credit rating.

Here, the borrower is the sender with project return distribution G_θ and private information μ. The lender plays the role of the receiver and sets an interest rate $r \in A \in \mathbf{R}$. A borrower with posterior ν that accepts a loan at rate r has a total payoff

$$\mathbf{E}_\nu \left[\int \max\{R - (1+r),\, 0\} dG_\theta(R) \right]$$

In case of no lending (including when the rating agency excludes the borrower), the payoff of the borrower is $R_0 \in (0,1)$ and the payoff of the lender is 0. Recall, $\hat{\mu}(r)$ is the type of borrower that is indifferent between investing at rate r and exercising her outside option.

$$\mathbf{E}_{\hat{\mu}(r)} \left[\int \max\{R - (1+r),\, 0\} dG_\theta(R) \right] = R_0$$

Thus, for belief β, the lender's payoff from offering an interest rate r is given by

$$\mathbf{E}_\beta \left[\int \min\{R-1,\, r\} dG_\theta(R) \,\middle|\, \nu \geq \hat{\mu}(r) \right] \cdot \mathrm{Prob}(\nu \geq \hat{\mu}(r))$$

A mechanism for the rating agency specifies whether the borrower is given access to credit. If the borrower is given access to credit, the mechanism provides information to the borrower leading to a posterior ν and recommends an interest rate r to the lender. The lender infers the distribution of posteriors ν of the borrower and agrees to lend at the recommended rate r if

$$\begin{aligned} &\mathbf{E}_\beta \left[\int \min\{R-1,\, r\} dG_\theta(R) \,\middle|\, \nu \geq \hat{\mu}(r) \right] \cdot \mathrm{Prob}(\nu \geq \hat{\mu}(r)) \\ &\qquad \geq \mathbf{E}_\beta \left[\int \min\{R-1,\, r'\} dG_\theta(R) \,\middle|\, \nu \geq \hat{\mu}(r') \right] \cdot \mathrm{Prob}(\nu \geq \hat{\mu}(r')) \end{aligned} \tag{12}$$

Let $\hat{r}(\nu)$ be the interest rate that makes the borrower with posterior ν indifferent between borrowing and exercising her outside option.

$$\mathbf{E}_\nu \left[\int \max\{R - (1+\hat{r}(\nu)),\, 0\} dG_\theta(R) \right] = R_0$$

For ease of exposure, we restrict attention to the following θ contingent return distributions

$$G_0 = \delta_{\overline{R}}$$

$$G_1 = \frac{\overline{R}}{R_1}\delta_{\overline{R}} + \left(1 - \frac{\overline{R}}{R_1}\right)\delta_0$$

Where δ_R is the Dirac mass at point R.
Additionally, we require the following relationship to hold

$$0 < R_0 < 1 < 1 + R_0 < \overline{R} < R_1 < \overline{R}\frac{\overline{R}}{\overline{R} - R_0}$$

The second inequality implies that the borrower can not fund the project with her safe option. The fourth inequality states that the total expected return from investing in the project is greater than the borrower's outside option. Note that $\hat{r}(\mu)$ is increasing thus any rate $r < \hat{r}(0)$ violates obedience Equation 12. In particular, for any incentive compatible and obedient direct mechanism, there is a payoff equivalent mechanism which recommends a rate $r \in [\hat{r}(0),\ \hat{r}(1)]$. The inequality $R_1 < \overline{R}\frac{\overline{R}}{\overline{R}-R_0}$ implies that $1 + \hat{r}(1) \leq \overline{R}$, thus we can rewrite the payoff of a borrower with posterior ν from borrowing at a rate r as following

$$\overline{R} - (1 + r) - \nu\left(1 - \frac{\overline{R}}{R_1}\right) \tag{13}$$

The payoff of a borrower with posterior ν from having access to credit is

$$\max\left\{R_0,\ \overline{R} - (1 + r) - \nu\left(1 - \frac{\overline{R}}{R_1}\right)\right\}$$

Participation Let $q_\theta(\mu)$ denote the probability that type μ gets access to credit when having riskiness θ, and let

$$\overline{q}(\mu) = \mu q_1(\mu) + (1 - \mu)q_0(\mu)$$

be the total probability that μ gets access to credit. As the borrower's payoff is R_0 when excluded from the mechanism, the borrower must earn a payoff greater than her outside option R_0 from following the mechanism's recommendation. Given any incentive compatible and obedient direct mechanism in which the borrower is recommended a rate r but chooses to exercise her outside option instead, we can construct a payoff equivalent mechanism in which the borrower always invests in the project when given access to credit. Such a full participation mechanism requires that the expected payoff for every type μ conditional on borrowing at a rate r is at least R_0:

$$\mu \cdot q_1(\mu) \geq \hat{\mu}(r) \cdot \overline{q}(\mu) \tag{14}$$

Incentive Compatibility The participation constraint in Equation 14 and the borrower's utility in Equation 13 allow us to use our characterization of incentive compatibility as the borrower's payoff is linear in her type along the equilibrium path.

Recall that the payoff of borrower with riskiness θ conditional on borrowing at rate r is given by Equation 13

$$u_\theta(r) = \overline{R} - (1+r) + \theta\left(1 - \frac{\overline{R}}{R_1}\right)(1+r)$$

This yields the following gross payoff

$$v(\mu) = \mu u_1(r) q_1(\mu) + (1-\mu) u_0(r) q_0(\mu)$$

The deviation payoff is

$$\begin{aligned} v(\mu' \mid \mu) &= \mu u_1(r) q_1(\mu') + (1-\mu) u_0(r) q_0(\mu') \\ &= v(\mu) + \mu u_1(r) \cdot (q_1(\mu') - q_1(\mu)) + (1-\mu) u_0(r) \cdot (q_0(\mu') - q_0(\mu)) \end{aligned}$$

and therefore the monotonicity condition requires that $u_1(r)q_1(\mu) - u_0(r)q_0(\mu)$ is weakly increasing. Equivalently, we can express this as

$$K(\mu) = u_1(r) \cdot q_1(\mu) - u_0(r) \cdot q_0(\mu) \quad \text{is weakly increasing in } \mu. \tag{15}$$

This condition, together with the envelope formula

$$U(\mu) = \int_0^\mu \left[u_1(r) \cdot q_1(\mu') - u_0(r) \cdot q_0(\mu')\right] d\mu' + U(0) \tag{16}$$

are necessary and sufficient for incentive compatibility.[4]

Individual Rationality Note that the indirect utility $U(\mu)$ must be greater than or equal to R_0 for all types. The rating agency's revenue is given by

$$\Pi(q) = \mathbf{E}v(\mu) - \mathbf{E}U(\mu)$$

[4] Like before, we are empowering the rating agency to enforce the loan when the lender accepts the recommended rate r.

By inspection of the revenue and convexity of U we conclude there must be some type $\mu \in [0,1]$ for which $U(\mu) = 0$. Moreover, this binding participation type is $\mu = 0$ as for any allocation q and for any rate $r \in [\hat{r}(0),\ \hat{r}(1)]$ we have the following

$$\mu u_1(r)q_1(0) + (1-\mu)u_0(r)q_0(0) + R_0 \cdot (1 - \mu q_1(0) - (1-\mu)q_0(0)) \\ \geq u_0(r)q_0(0) + R_0 \cdot (1 - q_0(0))$$

The above follows from noting that $u_0(r) \leq R_0 \leq u_1(r)$ for all $r \in [\hat{r}(0),\ \hat{r}(1)]$. It implies that any type $\mu' \neq 0$ gets at least as much utility as $\mu = 0$ from misreporting as type 0 and getting $q(0)$. If the smallest binding type is $\mu' > 0$, by convexity of U we get that $U(0) > U(\mu')$. This type μ' has an incentive to misreport as type 0, and thus incentive compatibility rules out $U(0) > R_0$.

In particular, for a revenue maximizing mechanism $U(0) = R_0$.

Obedience The payoff of the seller, with belief β, from lending at rate $r \in [\hat{r}(0),\ \hat{r}(1)]$ is given by the following

$$\mathbf{E}_\beta \left[r - \nu(1+r)\left(1 - \frac{\overline{R}}{R_1}\right) \,\middle|\, \nu \geq \hat{\mu}(r) \right] \cdot \text{Prob}(\nu \geq \hat{\mu}(r))$$

A central role in the study of obedience constraints (Equation 12) is played by the distribution of risk profiles that are iso-elastic on their support. These are distributions of risk profile such that the lender is indifferent between all rates r' for which $\hat{\mu}(r')$ is in the distribution's support. For a given r, a distribution in this class can be characterized by a mass point of size $x \in [0,1]$ at $\mu = \hat{\mu}(r)$ and a corresponding target mean $m(x)$. This target mean $m(x)$ is decreasing in x with $m(1) = \hat{\mu}(r) = \mathbf{E}H_1^r$ and $m(0) = \mathbf{E}H_0^r$. Where H_0^r is given by the following

$$H_0^r(\nu) = \begin{cases} 0 & \nu \leq \hat{\mu}(r) \\ 1 - \frac{J(\nu)}{J(\hat{\mu}(r))} & \hat{\mu}(r) < \nu < 1 \\ 1 & \nu = 1 \end{cases}$$

For $J(\nu) = 1 + g(\nu)\int_\nu^1 \frac{I(\mu')}{I(\nu)} d\mu'$, $I(\mu) = \exp\left(-\int_\mu^1 g(\mu')d\mu'\right)$ and $g(\mu) = \frac{\overline{R}-R_0}{\overline{R}-R_0-1}\frac{1-\frac{\overline{R}}{R_1}}{1-\mu\left(1-\frac{\overline{R}}{R_1}\right)}$.

In particular, $\psi(\mu) = \int_\mu^1 (\mu - \nu)dH_0^r(\nu)$ is the solution to the following ordinary differential equation with initial condition $\frac{d}{d\mu}\psi(\mu)\Big|_{\mu=\hat{\mu}(r)} = 1$

$$\frac{d}{d\mu}\psi(\mu) + g(\mu)\psi(\mu) = \lim_{\mu'\uparrow 1} H_0^r(\mu') \tag{17}$$

Additionally, in Section C.1 we show that H_x^r takes the following form

$$H_x^r(\nu) = \begin{cases} 0 & \nu < \hat{\mu}(r) \\ x & \hat{\mu}(r) \leq \nu \leq \widetilde{\mu}(x) \\ x + (1-x)H_0^{\hat{r}(\widetilde{\mu}(x))}(\nu) & \widetilde{\mu}(x) < \nu < 1 \\ 1 & \nu = 1 \end{cases}$$

Note that there is a bijection between the size of the mass point x at $\hat{\mu}(r)$ and the target mean $m(x) = \mathbf{E}H_x^r$. In particular we can identify a distribution H_x^p for each target mean $m \in [\hat{\mu}(r), \mathbf{E}H_0^r]$. Similar to Section 5, a distribution of the borrower's risk profile can be transformed into one that satisfies Equation 12 and Equation 14 if and only if it is a mean preserving contraction of some H_x^r. (see Proposition 4 in Section C.1).

We relax Equation 12 to the following conditions

$$\mathbf{E}\,\mathrm{marg}_{\Delta\theta}\,\beta(\cdot \mid r) = m \tag{18}$$

Where $m \in [\hat{\mu}(r), \mathbf{E}H_0^r]$. Corresponding to mean m, there is an iso-elastic risk profile H_x^r.

When the condition Equation 18 holds, and additionally, the CDF of $\mathrm{marg}_{\Delta\theta}\,\beta(\cdot \mid r)$ crosses H_x^r once and from below, then the former is a mean-preserving contraction of the latter. A necessary condition for this single crossing is

$$\mathrm{marg}_{\Delta\theta}\,\beta(\hat{\mu}(r) \mid r) \leq x \tag{19}$$

and the requirement that the support of $\mathrm{marg}_{\Delta\theta}\,\beta(\cdot \mid r)$ is in $[\hat{\mu}(r), 1]$ follows from Equation 14.

Relaxed Problem When we express the rating agency's profit as the difference between expected gross payoff and expected buyer utility, represent expected utility by the envelope formula with $U(0) = R_0$ and then proceed with the standard manipulations, we obtain the following objective function for the rating agency.

$$\Pi = \int_0^1 u_1(r)\left(\mu - \frac{1-F(\mu)}{f(\mu)}\right) q_1(\mu)dF(\mu) + \int_0^1 u_0(r)\left(1-\mu+\frac{1-F(\mu)}{f(\mu)}\right) q_0(\mu)dF(\mu) - R_0\int_0^1 \overline{q}(\mu)dF(\mu) \quad (20)$$

Our relaxed problem is to choose $q_\theta(\cdot)$ to maximize this profit function subject to monotonicity (Equation 15), participation (Equation 14), and relaxed obedience (Equation 18 and Equation 19) for some $m \in [\hat{\mu}(r), \mathbf{E}H_0^r]$.

Proposition 3. *If $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing, then for any r one of the following $q(\mu) = (q_1(\mu), q_0(\mu))$ is a solution to the relaxed problem*

$$q(\mu) = \begin{cases} (0,0) & \text{if} \quad \mu \leq \mu_0 \\ \left(\alpha_r^{-1}(\mu_0), 1\right) & \text{if} \quad \mu \in [\mu_0, \mu_1) \\ (1,1) & \text{if} \quad \mu \in [\mu_1, \overline{\mu}) \\ (1,0) & \quad \text{otherwise} \end{cases} \quad \text{or} \quad q(\mu) = \begin{cases} (0,0) & \text{if} \quad \mu \leq \mu_0 \\ \left(\alpha_r^{-1}(\mu_0), 1\right) & \text{if} \quad \mu \in [\mu_0, \overline{\mu}) \\ \left(\alpha_r^{-1}(\mu_0) - \frac{u_1(r)}{u_0(r)}, 0\right) & \quad \text{otherwise} \end{cases}$$

for $\hat{\mu}(r) < \mu_0$ and $\alpha_r(\mu_0) = \frac{1-\hat{\mu}(r)}{\hat{\mu}(r)}\frac{\mu_0}{1-\mu_0}$.

$$q(\mu) = \begin{cases} (0,0) & \text{if} \quad \mu \leq \mu_0 \\ (1, \alpha_r(\mu_0)) & \text{if} \quad \mu \in [\mu_0, \overline{\mu}) \\ (1,0) & \quad \text{otherwise} \end{cases} \quad \text{or} \quad q(\mu) = \begin{cases} (0,0) & \text{if} \quad \mu \leq \mu_0 \\ (1, \alpha_r(\mu_0)) & \text{if} \quad \mu \in [\mu_0, \overline{\mu}) \\ \left(1 - \alpha_r(\mu_0)\frac{u_1(r)}{u_0(r)}, 0\right) & \quad \text{otherwise} \end{cases}$$

for $\mu_0 \leq \hat{\mu}(r)$ and $\alpha_r(\mu_0) = \frac{1-\hat{\mu}(r)}{\hat{\mu}(r)}\frac{\mu_0}{1-\mu_0}$.

7 Related Literature

Corrao (2023) and Lizzeri (1999) study variations of an information design problem with incentive constraints but in their model the sender is already perfectly informed. The role of the intermediary is to turn the sender's soft information into information that is verifiable to the receiver. In Ali et al. (2020) the intermediary is generating new information but the sender has no private information *ex ante* and so there are no screening frictions. Bergemann et al. (2018) study a problem like ours where the information designer produces new information for an *ex ante* privately informed agent. However, in Bergemann et al. (2018) the sender herself takes the action *ex post*, there is no third party/receiver.

Three concurrent and independent papers are closest to ours. Celik and Strausz (2025) and Weksler and Zik (2025) both study a buyer-seller interaction like our digital platform and Mäkimattila et al. (2024) study a version of our college admissions example. All three papers impose a restriction on the intermediary that the test chosen by the sender must be revealed to the receiver. By contrast we give the intermediary full flexibility in designing the information structure for the sender and the receiver. This ability to pool is valuable to the intermediary and shapes the optimal mechanism. As a benchmark Mäkimattila et al. (2024) do also examine the fully flexible case and derive the optimal mechanism in a case similar to our item 1.

References

Ali, Nageeb, Nima Haghpanah, Xiao Lin, and Ron Siegel, "How to Sell Hard Information," *Quarterly Journal of Economics*, 2020, *135* (4), 1757–1811.

Barvinok, Alexander, *A course in convexity*, Vol. 54, American Mathematical Society, 2025.

Bergemann, Dirk, Alessandro Bonatti, and Alex Smolin, "The Design and Price of Information," *American Economic Review*, 2018, *108* (1), 1–48.

Celik, Gorkem and Roland Strausz, "Informative Certification: Screening vs. Acquisition," Technical Report 525, CRC TRR 190 Rationality and Competition 2025.

Chopra, Hershdeep and Jeffrey Ely, "Interim Information Design," 2025. Working paper.

Corrao, Roberto, "Mediation Markets," 2023. Working Paper.

Dubins, Lester E., "On extreme points of convex sets," *Journal of Mathematical Analysis and Applications*, 1960, *1* (1), 1–3.

Kamenica, Emir and Matthew Gentzkow, "Bayesian Persuasion," *American Economic Review*, 2011, *101* (6), 2590–2615.

Lizzeri, Alessandro, "Certification Intermediaries," *RAND Journal of Economics*, 1999, *30* (2), 214–231.

Mäkimattila, Mikael, Yucheng Shang, and Ryo Shirakawa, "The Design and Price of Certification," 2024. Working paper.

Roesler, Anne-Katrin and Balázs Szentes, "Buyer-optimal learning and monopoly pricing," *American Economic Review*, 2017, *107* (7), 2072–2080.

Weksler, Ran and Boaz Zik, "Monopolistic Screening in Certification Markets," 2025. Working paper.

A College Admissions

A.1 Geometry of Monotone Tests

The slope of the indirect utility is given by $K(\mu) = q_1(\mu) - q_0(\mu)$. Substituting this into Equation 2 gives us

$$\Pi = \mathbf{E}\left[q_1(\mu)\right] - \mathbf{E}\left[K(\mu)\left(1 - \mu + \frac{1 - F(\mu)}{f(\mu)}\right)\right]$$

The obedience constraint is given by

$$\mathbf{E}\left[\mu q_1(\mu)\right] \geq \frac{p}{1-p}\mathbf{E}\left[(1-\mu)q_0(\mu)\right]$$

As we argued in the text the obedience constraint must bind at an optimum. We can express the binding constraint in terms of the slope function $K(\mu)$ as follows.

$$\mathbf{E}\left[(1-\mu)K(\mu)\right] = \mathbf{E}\left[\left(1 - \frac{\mu}{p}\right)q_1(\mu)\right]$$

Putting together,

$$\Pi = \mathbf{E}\left[\frac{\mu}{p}q_1(\mu)\right] - \mathbf{E}\left[K(\mu)\frac{1 - F(\mu)}{f(\mu)}\right] \tag{21}$$

If K is such that there exists some incentive compatible and obedient test (q_1, q_0) for which $K(\mu) = q_1(\mu) - q_0(\mu)$, the following holds

$$0 \leq K(\mu) \leq q_1(\mu) \leq 1 \tag{22}$$

and

$$\mathbf{E}\left[(1-\mu)K(\mu)\right] \geq \int_0^1 \left(1 - \frac{\mu}{p}\right) q_1(\mu)dF(\mu) \tag{23}$$

We call a slope K feasible if it satisfies Equation 22 and Equation 23 for some test allocation q_1. Define the following class of threshold allocations

$$(q_1^K(\mu),\, q_0^K(\mu)) := \begin{cases} (K(\mu),\, 0) & \mu < \mu^K \\ (1,\, 1 - K(\mu)) & \mu \geq \mu^K \end{cases}$$

For some $\mu^K \in [0, 1]$.

Lemma 1. *For given p and feasible $K(\mu)$, the optimal allocation is $q_1 = q_1^K$ for some μ^K.*

Proof. It is clear from Equation 21 and Equation 23 that the optimum requires $q_1(\mu) = 1$ for all $\mu \geq p$.

Consider for contradiction that there is some interval of types $[\mu_0, \overline{\mu})$ such that $K(\mu) < q_1(\mu) < 1$. It's sufficient to consider $\overline{\mu} \leq p$. Pick $\delta > 0$ such that $\overline{\mu} - \mu_0 \geq 2\delta$. We can improve the revenue by increasing q_1 on $[\overline{\mu} - \delta, \overline{\mu})$ by some $\varepsilon_1 > 0$ and by reducing q_1 on $[\mu_0, \mu_0 + \delta)$ by $\varepsilon_0 > 0$ where

$$\varepsilon_1 = \varepsilon_0 \frac{\int_{\mu_0}^{\mu_0+\delta} \left(1 - \frac{\mu}{p}\right) dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} \left(1 - \frac{\mu}{p}\right) dF(\mu)}$$

ensuring that Equation 23 is maintained.

The revenue from the new allocation is greater than the old allocation if the following holds:

$$\frac{\int_{\mu_0}^{\mu_0+\delta} \left(1 - \frac{\mu}{p}\right) dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} \left(1 - \frac{\mu}{p}\right) dF(\mu)} \geq \frac{\int_{\mu_0}^{\mu_0+\delta} \frac{\mu}{p} dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} \frac{\mu}{p} dF(\mu)}$$

$$\iff \frac{\int_{\mu_0}^{\mu_0+\delta} dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} dF(\mu)} \geq \frac{\int_{\mu_0}^{\mu_0+\delta} \mu dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} \mu dF(\mu)}$$

The last inequality follows from our choice of δ.

Now consider q_1 such that there are intervals $I_1 < I_2$ where $q_1(\mu) = K(\mu)$ for $\mu \in I_2$ and $q_1(\mu) = 1$ for $\mu \in I_1$. It suffices to consider $\sup(I_2) \leq p$ by the previous arguments. We construct an improvement similar to the above by slightly increasing q_1 on I_2 and reducing q_1 on I_1 to preserve the inequality in Equation 23. The revenue of this improvement is greater than the old allocation if the following holds:

$$\frac{\int_{I_1} dF(\mu)}{\int_{I_2} dF(\mu)} \geq \frac{\int_{I_1} \mu dF(\mu)}{\int_{I_2} \mu dF(\mu)}$$

This is implied by $\frac{\inf(I_2)}{\sup(I_1)} \geq 1$. □

By Lemma 1 we can, without loss, focus on allocations that have non-decreasing admission probability q_1. Let Λ be the set of all such allocations which are also incentive-compatible and have indirect utility with $U(0) = 0$. When viewed as a subset of the topological vector space (TVS) $\mathbb{L}_\infty([0,1] \to \mathbb{R}^2)$, the set Λ is algebraically compact and convex.[5] Moreover, its extreme points are such that for some $0 \le \mu_0 \le \overline{\mu} \le 1$

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu \in [0, \mu_0) \\ (1,1) & \mu \in [\mu_0, \overline{\mu}) \\ (1,0) & \mu \in [\overline{\mu}, 1] \end{cases}$$

The set of such allocations is $\text{ex}(\Lambda)$.

Another consequence of Lemma 1 is that, fixing μ^K, the designer's objective can be represented as maximizing a linear functional, where the choice variable is the slope of the indirect utility, a monotone function! Thus, the slope K for an extreme test allocation is a non-decreasing single-step function. By virtue of a linear objective and a single linear obedience constraint, an optimal allocation can be represented as a convex combination of at most two elements of $\text{ex}(\Lambda)$ (see Dubins (1960)). Thus, the optimal revenue is achieved by a mechanism where the slope of the indirect utility is a non-decreasing two-step function. In particular, the optimal test can be found among the ones with the following structure

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu \in [0, \mu_0) \\ (1,1) & \mu \in [\mu_0, \mu_1) \\ (1,\alpha) & \mu \in [\mu_1, \overline{\mu}) \\ (1,0) & \mu \in [\overline{\mu}, 1] \end{cases} \quad \text{or} \quad (q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu \in [0, \mu_0) \\ (1-\alpha,0) & \mu \in [\mu_0, \mu_1) \\ (1,\alpha) & \mu \in [\mu_1, \overline{\mu}) \\ (1,0) & \mu \in [\overline{\mu}, 1] \end{cases}$$

The proposition then establishes sufficient conditions on the prior F under which 1) the optimal allocation is described by a single extreme point, 2) the optimal allocation is described by convex combination of two extreme points.

A.2 Proof of Proposition 1

Proof of Proposition 1. First, we note that a test such that $q_0 = 0$ is never optimal; this can be seen as the obedience constraint is always slack for such tests.
To prove item 1, we show by contradiction that a convex combination of two distinct tests can be improved. By Lemma 1 we can restrict attention to tests where

[5] A set Λ in a TVS is algebraically compact if the intersection of the set Λ with a line is always algebraically closed and bounded, see Barvinok (2025).

$q_1(\mu) = q_0(\mu)$ implies that either $q_1(\mu) = 0$ or $q_1(\mu) = 1$. First consider as a test (q_1, q_0) such that there exists $0 \le \mu_0 \le \mu_1 < \overline{\mu} \le 1$ and

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu \in [0, \mu_0) \\ (1,1) & \mu \in [\mu_0, \mu_1) \\ (1,\alpha) & \mu \in [\mu_1, \overline{\mu}) \\ (1,0) & \mu \in [\overline{\mu}, 1] \end{cases}$$

We will construct a more profitable test $q' = (q_1', q_0')$ such that $q_1' = q_1$ for all μ and $q_0 = q_0'$ on the complement of $[\mu_1, \overline{\mu})$. On $[\mu_1, \mu_1 + \varepsilon)$ set $q_0' = 1$ for some small $\varepsilon > 0$ and on $[\mu_1 + \varepsilon, \overline{\mu})$ set $q_0' = (1-\delta)\alpha$ for small enough $\delta > 0$. Now we will argue that whenever $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is decreasing there exist $\varepsilon > 0$ and $\delta > 0$ such that q' is feasible and earns higher revenue. Revenue increases by

$$(1-\alpha) \int_{\mu_1}^{\mu_1+\varepsilon} \left[1 - \mu + \frac{1-F(\mu)}{f(\mu)}\right] dF(\mu) - \delta\alpha \int_{\mu_1+\varepsilon}^{\overline{\mu}} \left[1 - \mu + \frac{1-F(\mu)}{f(\mu)}\right] dF(\mu)$$

and the slack in the obedience constraint increases by

$$-(1-\alpha) \int_{\mu_1}^{\mu_1+\varepsilon} \frac{(1-\mu)p}{1-p} dF(\mu) + \delta\alpha \int_{\mu_1+\varepsilon}^{\overline{\mu}} \frac{(1-\mu)p}{1-p} dF(\mu)$$

For a fixed $\varepsilon > 0$ there exist a $\delta > 0$ such that both are positive if and only if

$$\frac{\int_{\mu_1}^{\mu_1+\varepsilon}[1-\mu+\frac{1-F(\mu)}{f(\mu)}]dF(\mu)}{\int_{\mu_1+\varepsilon}^{\overline{\mu}}[1-\mu+\frac{1-F(\mu)}{f(\mu)}]dF(\mu)} > \frac{\int_{\mu_1}^{\mu_1+\varepsilon} \frac{(1-\mu)p}{1-p} dF(\mu)}{\int_{\mu_1+\varepsilon}^{\overline{\mu}} \frac{(1-\mu)p}{1-p} dF(\mu)} = \frac{\int_{\mu_1}^{\mu_1+\varepsilon}(1-\mu)dF(\mu)}{\int_{\mu_1+\varepsilon}^{\overline{\mu}}(1-\mu)dF(\mu)}$$

and this inequality holds if and only if

$$\frac{\int_{\mu_1}^{\mu_1+\varepsilon} \frac{1-F(\mu)}{f(\mu)} dF(\mu)}{\int_{\mu_1+\varepsilon}^{\overline{\mu}} \frac{1-F(\mu)}{f(\mu)} dF(\mu)} > \frac{\int_{\mu_1}^{\mu_1+\varepsilon}(1-\mu)dF(\mu)}{\int_{\mu_1+\varepsilon}^{\overline{\mu}}(1-\mu)dF(\mu)} \tag{24}$$

For $\varepsilon = 0$ both sides are zero. Now suppose $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is decreasing in μ. Then

$$\frac{1-F(\mu_1)}{(1-\mu_1)f(\mu_1)} \ge \int_{\mu_1}^{\overline{\mu}} \frac{1-F(\mu)}{(1-\mu)f(\mu)} dF(\mu) \ge \frac{\int_{\mu_1}^{\overline{\mu}} \frac{1-F(\mu)}{f(\mu)} dF(\mu)}{\int_{\mu_1}^{\overline{\mu}}(1-\mu)dF(\mu)}$$

or

$$\frac{\frac{1-F(\mu_1)}{f(\mu_1)}}{\int_{\mu_1}^{\overline{\mu}} \frac{1-F(\mu)}{f(\mu)} dF(\mu)} \geq \frac{1-\mu_1}{\int_{\mu_1}^{\overline{\mu}} (1-\mu) dF(\mu)}$$

and the left-hand side is the derivative of the left-hand side in Equation 24 while the right-hand side is the derivative of the right-hand side in Equation 24, both at $\varepsilon = 0$. This guarantees that Equation 24 holds on a neighborhood of $\varepsilon = 0$.

Now consider some test (q_1, q_0) which has the following form

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu \in [0, \mu_0) \\ (1-\alpha, 0) & \mu \in [\mu_0, \mu_1) \\ (1, \alpha) & \mu \in [\mu_1, \overline{\mu}) \\ (1,0) & \mu \in [\overline{\mu}, 1) \end{cases}$$

If $\mu_0 = \mu_1$ and $\mu_1 < \overline{\mu}$ then previous argument shows that (q_1, q_0) is not optimal. When $\mu_0 < \mu_1$ and $\mu_1 < \overline{\mu}$, then we can use the same ideas to construct a profitable deviation, let (q_1', q_0') such that $q_1' = q_1$ for all μ and $q_0 = q_0'$ on $([\mu_0, \overline{\mu})^C$. On $[\mu_0, \mu_0 + \varepsilon)$ set $q_0' = 1 - \alpha$ for some small $\varepsilon > 0$ and on $[\mu_1, \overline{\mu})$ set $q_0' = (1-\delta)\alpha$ for small enough $\delta > 0$. Now we will argue that whenever $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is decreasing there exist $\varepsilon > 0$ and $\delta > 0$ such that the proposed deviation is feasible and leads to a higher revenue. The increase in revenue and obedience slack are

$$(1-\alpha) \int_{\mu_0}^{\mu_0+\varepsilon} [1 - \mu + \frac{1-F(\mu)}{f(\mu)}] dF(\mu) - \delta\alpha \int_{\mu_1}^{\overline{\mu}} [1 - \mu + \frac{1-F(\mu)}{f(\mu)}] dF(\mu)$$

and

$$-(1-\alpha) \int_{\mu_0}^{\mu_0+\varepsilon} \frac{(1-\mu)p}{1-p} dF(\mu) + \delta\alpha \int_{\mu_1}^{\overline{\mu}} \frac{(1-\mu)p}{1-p} dF(\mu),$$

respectively. For a fixed $\varepsilon > 0$ there exist a $\delta > 0$ such that both are positive if and only if

$$\frac{\int_{\mu_0}^{\mu_0+\varepsilon} \frac{1-F(\mu)}{f(\mu)} dF(\mu)}{\int_{\mu_1}^{\overline{\mu}} \frac{1-F(\mu)}{f(\mu)} dF(\mu)} > \frac{\int_{\mu_0}^{\mu_0+\varepsilon} 1 - \mu \, dF(\mu)}{\int_{\mu_1}^{\overline{\mu}} 1 - \mu \, dF(\mu)}$$

which can be shown by a similar derivation as above to be true for small enough $\varepsilon > 0$. We have established item 1 that $(q_1, q_0) \in \text{ex}(\Lambda)$.

To prove item 2, consider some $(q_1, q_0) \in \text{ex}(\Lambda)$ and $0 \leq \mu_0 < \overline{\mu} < 1$ such that

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu \in [0, \mu_0) \\ (1,1) & \mu \in [\mu_0, \overline{\mu}) \\ (1,0) & \mu \in [\overline{\mu}, 1] \end{cases}$$

We will show that there exist tests (q_1', q_0') and (q_1'', q_0'') such that a mixture of these is feasible and more profitable than $(q_1(\mu), q_0(\mu))$. To this end for some $\varepsilon > 0$ define

$$(q_1'(\mu), q_0'(\mu)) := \begin{cases} (0,0) & \mu \in [0, \mu_0) \\ (1,1) & \mu \in [\mu_0, \overline{\mu} - \varepsilon) \\ (1,0) & \mu \in [\overline{\mu} - \varepsilon, 1] \end{cases}$$

$$(q_1''(\mu), q_0''(\mu)) := \begin{cases} (0,0) & \mu \in [0, \mu_0) \\ (1,1) & \mu \in [\mu_0, \overline{\mu} + \varepsilon) \\ (1,0) & \mu \in [\overline{\mu} + \varepsilon, 1] \end{cases}$$

Now define a "Revenue" and an "Obedience" function on $[\mu_0, 1]$;

$$R(\mu) := \int_{\mu_0}^{\mu} dF(\widetilde{\mu}) + \int_{\mu}^{1} [\widetilde{\mu} - \frac{1 - F(\widetilde{\mu})}{f(\widetilde{\mu})}] \, dF(\widetilde{\mu})$$

$$Q(\mu) := \int_{\mu_0}^{\mu} \frac{\widetilde{\mu} - p}{1 - p} dF(\widetilde{\mu}) + \int_{\mu}^{1} \widetilde{\mu} \, dF(\widetilde{\mu})$$

Note that R and $-Q$ are differentiable and increasing. Let

$$\alpha' := \inf\{\alpha \in [0,1] | \alpha R(\overline{\mu} - \varepsilon) + (1 - \alpha) R(\overline{\mu} + \varepsilon) = R(\overline{\mu})\}$$

$$\alpha'' := \inf\{\alpha \in [0,1] | \alpha Q(\overline{\mu} - \varepsilon) + (1 - \alpha) Q(\overline{\mu} + \varepsilon) = Q(\overline{\mu})\}$$

Also define the corresponding types $\mu' := \overline{\mu} + (1 - 2\alpha')\varepsilon$ and $\mu'' := \overline{\mu} + (1 - 2\alpha'')\varepsilon$. In particular, we get that

$$\alpha' = \frac{R(\overline{\mu} + \varepsilon) - R(\overline{\mu})}{R(\overline{\mu} + \varepsilon) - R(\overline{\mu} - \varepsilon)} \quad \text{and} \quad \alpha'' = \frac{Q(\overline{\mu}) - Q(\overline{\mu} + \varepsilon)}{Q(\overline{\mu} - \varepsilon) - Q(\overline{\mu} + \varepsilon)}$$

The proof follows by showing that $\mu' < \mu''$, for which it is sufficient to show $\alpha' > \alpha''$.

$$\frac{R(\overline{\mu} + \varepsilon) - R(\overline{\mu})}{R(\overline{\mu} + \varepsilon) - R(\overline{\mu} - \varepsilon)} > \frac{Q(\overline{\mu}) - Q(\overline{\mu} + \varepsilon)}{Q(\overline{\mu} - \varepsilon) - Q(\overline{\mu} + \varepsilon)}$$

$$\iff \frac{\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon}[1-\mu+\frac{1-F(\mu)}{f(\mu)}]dF(\mu)}{\int_{\overline{\mu}-\varepsilon}^{\overline{\mu}+\varepsilon}[1-\mu+\frac{1-F(\mu)}{f(\mu)}]dF(\mu)} > \frac{\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon}(1-\mu)dF(\mu)}{\int_{\overline{\mu}-\varepsilon}^{\overline{\mu}+\varepsilon}(1-\mu)dF(\mu)}$$

$$\iff \frac{\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon}\frac{1-F(\mu)}{f(\mu)}dF(\mu)}{\int_{\overline{\mu}-\varepsilon}^{\overline{\mu}+\varepsilon}\frac{1-F(\mu)}{f(\mu)}dF(\mu)} > \frac{\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon}(1-\mu)dF(\mu)}{\int_{\overline{\mu}-\varepsilon}^{\overline{\mu}+\varepsilon}(1-\mu)dF(\mu)}$$

$$\iff \frac{\int_{\overline{\mu}-\varepsilon}^{\overline{\mu}}\frac{1-F(\mu)}{f(\mu)}dF(\mu)}{\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon}\frac{1-F(\mu)}{f(\mu)}dF(\mu)} < \frac{\int_{\overline{\mu}-\varepsilon}^{\overline{\mu}}(1-\mu)dF(\mu)}{\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon}(1-\mu)dF(\mu)}$$

Whenever $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-decreasing the required inequality holds as

$$\frac{\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon}\frac{1-F(\mu)}{f(\mu)}dF(\mu)}{\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon}(1-\mu)dF(\mu)} \geq \frac{1-F(\overline{\mu})}{(1-\overline{\mu})f(\overline{\mu})} \geq \frac{\int_{\overline{\mu}-\varepsilon}^{\overline{\mu}}(\frac{1-F(\mu)}{f(\mu)})dF(\mu)}{\int_{\overline{\mu}-\varepsilon}^{\overline{\mu}}(1-\mu)dF(\mu)}$$

Recall that any test can be written as a convex combination of at most two extreme points. The above argument shows that if $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-decreasing, then the optimal solution is represented by a convex combination of extreme points for which either $\overline{\mu} = 1$ or $\overline{\mu} = \mu_0$. Using Lemma 1, the optimal test can be expressed in the following form:

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu \in [0, \mu_0) \\ (1-\alpha, 0) & \mu \in [\mu_0, \overline{\mu}) \\ (1, \alpha) & \mu \in [\overline{\mu}, 1] \end{cases}$$

□

B Digital Platform

B.1 Relaxed Problem

Recall that $K(\mu) = (1-p)q_1(\mu) + pq_0(\mu)$ is the slope of the indirect utility. We may substitute into the objective in Equation 11 to obtain

$$\Pi = \int_0^1 \mu(1-p)q_1(\mu)dF(\mu) - \int_0^1 p\,(1-\mu)\,q_0(\mu)dF(\mu) \\ - \int_0^1 K(\mu)\left(\frac{1-F(\mu)}{f(\mu)}\right)dF(\mu)$$

For any incentive-compatible and obedient mechanism q there is a corresponding monotone slope function $K(\mu)$ taking values in $[0,1]$ and a target *platform mean* m defined by

$$\int_0^1 \mu q_1(\mu)dF(\mu) = \frac{m}{1-m}\int_0^1 (1-\mu)q_0(\mu)dF(\mu)$$

or

$$\int_0^1 \left[1-p-\mu\left(1-\frac{p}{m}\right)\right]q_1(\mu)dF(\mu) = \int_0^1 (1-\mu)K(\mu)dF(\mu) \tag{25}$$

Substituting again into the objective and re-arranging we arrive at:

$$\Pi = \int_0^1 \mu\left(1-\frac{p}{m}\right)q_1(\mu)dF(\mu) - \int_0^1 \frac{1-F(\mu)}{f(\mu)}K(\mu)dF(\mu) \tag{26}$$

We have organized the obective function in a way that isolates the gains from increasing $q_1(\mu)$ from the costs of increasing the slope $K(\mu)$. In the background we can adjust $q_0(\mu)$ to maintain a given slope $K(\mu)$ and target mean m provided $q_1(\mu) \in [0,1]$ satisfies

$$\frac{K(\mu)-p}{1-p} \leq q_1(\mu) \leq \frac{K(\mu)}{1-p}. \tag{27}$$

The first inequality ensures $q_0(\mu) \geq 0$ and the second ensures $q_0(\mu) \leq 1$. We can also express the participation constraint Equation 4 in terms of $K(\mu)$ as follows

$$q_1(\mu) \geq \frac{(1-\mu)K(\mu)}{1-p}. \tag{28}$$

We will say that a slope function $K(\mu)$ is *feasible* with respect to a target mean m if there exists an allocation $q_1(\mu)$ which satisfies Equation 25, Equation 27, and Equation 28.

B.2 Optimal Allocation for Fixed K

We can approach the problem by first finding the optimal q_1 for a given target mean m and feasible $K(\mu)$, and then optimizing the latter.

Define the following class of bang-bang allocations.

$$q_1^K(\mu) := \begin{cases} \max\left\{\frac{(1-\mu)K(\mu)}{1-p}, \frac{K(\mu)-p}{1-p}\right\} & \mu < \mu^K \\ \min\left\{1, \frac{K(\mu)}{1-p}\right\} & \mu \geq \mu^K \end{cases}$$

For some $\mu^K \in [0,1]$.

Lemma 2. *For given m and feasible $K(\mu)$, the optimal allocation is $q_1 = q_1^K$ for some μ^K.*

Proof. Consider any q_1 for which there is an interval of types, $[\mu_0, \overline{\mu}]$ such that $\max\left\{\frac{(1-\mu)K(\mu)}{1-p}, \frac{K(\mu)-p}{1-p}\right\} < q_1(\mu) < \min\left\{1, \frac{K(\mu)}{1-p}\right\}$. Pick $\delta > 0$ such that $\overline{\mu} - \mu_0 \geq 2\delta$. We can improve the revenue by increasing q_1 on $[\overline{\mu} - \delta, \overline{\mu})$ by some $\varepsilon_1 > 0$ and by reducing q_1 on $[\mu_0, \mu_0 + \delta)$ by $\varepsilon_0 > 0$ where

$$\varepsilon_1 = \varepsilon_0 \frac{\int_{\mu_0}^{\mu_0+\delta} \left(1 - p - \mu\left(1 - \frac{p}{m}\right)\right) dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} \left(1 - p - \mu\left(1 - \frac{p}{m}\right)\right) dF(\mu)}$$

ensuring that the target mean, i.e. Equation 25 is maintained. This adjustment doesn't violate the constraint in Equation 10 as it weakly decreases the size of any point mass at p without changing the target mean.

The revenue from the new allocation is greater than the old allocation if the following holds:

$$\frac{\int_{\mu_0}^{\mu_0+\delta} \left(1 - p - \mu\left(1 - \frac{p}{m}\right)\right) dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} \left(1 - p - \mu\left(1 - \frac{p}{m}\right)\right) dF(\mu)} \geq \frac{\int_{\mu_0}^{\mu_0+\delta} \mu\left(1 - \frac{p}{m}\right) dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} \mu\left(1 - \frac{p}{m}\right) dF(\mu)}$$

$$\iff \frac{\int_{\mu_0}^{\mu_0+\delta} (1-p)\, dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} (1-p)\, dF(\mu)} \geq \frac{\int_{\mu_0}^{\mu_0+\delta} \mu\left(1 - \frac{p}{m}\right) dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} \mu\left(1 - \frac{p}{m}\right) dF(\mu)}$$

$$\iff \frac{\int_{\mu_0}^{\mu_0+\delta} dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} dF(\mu)} \geq \frac{\int_{\mu_0}^{\mu_0+\delta} \mu dF(\mu)}{\int_{\overline{\mu}-\delta}^{\overline{\mu}} \mu dF(\mu)}$$

The last inequality follows from our choice of δ.

Now consider q_1 such that there are intervals $I_1 < I_2$ where $q_1(\mu) = \max\left\{\frac{(1-\mu)K(\mu)}{1-p}, \frac{K(\mu)-p}{1-p}\right\}$ for $\mu \in I_2$ and $q_1(\mu) = \min\left\{1, \frac{K(\mu)}{1-p}\right\}$ for $\mu \in I_1$. We construct an improvement similar to the above by slightly increasing q_1 on I_2 while reducing q_1 on I_1 to keep the mean constraint binding. Finally, note that this improvement introduces more slack to Equation 10 . The revenue of this improvement is greater than the old allocation if the following holds:

$$\frac{\int_{I_1} dF(\mu)}{\int_{I_2} dF(\mu)} \geq \frac{\int_{I_1} \mu dF(\mu)}{\int_{I_2} \mu dF(\mu)}$$

This is implied by $\frac{\inf(I_2)}{\sup(I_1)} \geq 1$. □

B.3 Optimal Choice of K

Notice that the monotonicity of $K(\mu)$ implies that there exists μ_0 such that $\frac{(1-\mu)K(\mu)}{1-p} \geq \frac{K(\mu)-p}{1-p}$ for all $\mu \leq \mu_0$ and $\frac{(1-\mu)K(\mu)}{1-p} < \frac{K(\mu)-p}{1-p}$ for all $\mu > \mu_0$. In particular the participation constraint in Equation 28 binds only for types $\mu \leq \mu_0$ and all mass x at the point p comes from these types. The next lemma states that $x = 0$ for a solution to the relaxed problem.

Lemma 3. *The solution to the relaxed problem has $K(\mu) = 0$ for all $\mu < \mu_0$.*

Proof. Consider a feasible allocation with target mean m and mass size x. Suppose there is a non-empty interval $[\underline{\mu}, \mu_0)$ consiting of types μ for which $q_1(\mu) = \frac{(1-\mu)K(\mu)}{1-p} > 0$. Then $x > 0$ and $m \leq \mathbf{E}H_0^p$. Consider a new allocation which reduces $K(\mu)$ slightly at all points in $[\underline{\mu}, \mu_0)$ while keeping q_1 unchanged. The new allocation increases the mean slightly (by Equation 9) to say $m' > m$ and eliminates any mass point (Equation 4 will be slack). For m' close enough to m the new allocation is feasible for the relaxed problem with target mean m' and mass $x = 0$. That is, Equation 9 and Equation 10 are satisfied for m' and $x = 0$ because $\mathbf{E}H_0^p > \mathbf{E}H_x^p$.

By inspection of the objective in Equation 26 this is an improvement. Therefore a solution to the relaxed problem must have $K(\mu) = 0$ for all $\mu < \mu_0$. □

Lemma 2 and Lemma 3 imply that a solution to the relaxed problem can be described by μ_0 and two further thresholds $\mu_0 \leq \mu_1 \leq \overline{\mu}$ such that

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu < \mu_0 \\ \left(\frac{K(\mu)-p}{1-p}, 1\right) & \mu \in [\mu_0, \mu_1) \\ \left(\frac{K(\mu)}{1-p}, 0\right) & \mu \in [\mu_1, \overline{\mu}) \\ \left(1, \frac{1}{p}\left(K(\mu) - (1-p)\right)\right) & \mu > \overline{\mu} \end{cases} \tag{29}$$

Lemma 4. *A solution to the relaxed problem has a threshold $\mu' \geq \overline{\mu}$ such that*

1. $K(\mu) = 1$ *for all* $\mu \geq \mu'$.
2. $q_0(\mu)$ *is constant on* $[\mu_0, \mu')$ *and equal to* 0 *or* 1.

Proof. We first show that any candidate solution can be weakly improved by one that has $q_0(\mu) \in \{0, 1\}$ for all $\mu \in [\overline{\mu}, 1]$. If the candidate allocation does not already have that property then we construct a new allocation q' such that on $q'(\mu) = q(\mu)$ for $\mu \in [\overline{\mu}, 1]^C$. On the interval $[\overline{\mu}, \overline{\mu} + \varepsilon_0)$ let $q_0'(\mu) = q_0(\overline{\mu})$, on the interval $[1-\varepsilon_1, 1]$ let $q_0'(\mu) = 1$, and $q_0'(\mu) = q_0(\mu)$ for $\mu \in [\overline{\mu} + \varepsilon_0, 1 - \varepsilon_1)$. Where

$$\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon_0} (1-\mu)(q_0(\mu) - q_0(\overline{\mu}))dF(\mu) - \int_{1-\varepsilon_1}^{1} (1-\mu)(1-q_0(\mu))dF(\mu) = 0$$

As long as $\overline{\mu} + \varepsilon_0 \leq 1 - \varepsilon_1$, the allocation q' is well-defined and has the same target mean as q. Consider the difference in revenue between q and q':

$$\int_{\overline{\mu}}^{\overline{\mu}+\varepsilon_0} \left(1-\mu+\frac{1-F(\mu)}{f(\mu)}\right)(q_0(\mu) - q_0(\overline{\mu}))dF(\mu) - \int_{1-\varepsilon_1}^{1} \left(1-\mu+\frac{1-F(\mu)}{f(\mu)}\right)(1-q_0(\mu))dF(\mu)$$

$$= \int_{\overline{\mu}}^{\overline{\mu}+\varepsilon_0} \frac{1-F(\mu)}{f(\mu)}(q_0(\mu) - q_0(\overline{\mu}))dF(\mu) - \int_{1-\varepsilon_1}^{1} \frac{1-F(\mu)}{f(\mu)}(1-q_0(\mu))dF(\mu)$$

When $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing the change in revenue is positive as

$$\int_{1-\varepsilon_1}^{1} \frac{1-F(\mu)}{f(\mu)}(1-q_0(\mu))dF(\mu) \leq \frac{1-F(1-\varepsilon_1)}{\varepsilon_1 f(1-\varepsilon_1)} \int_{1-\varepsilon_1}^{1} (1-\mu)(1-q_0(\mu))dF(\mu)$$

$$\implies \int_{1-\varepsilon_1}^{1} \frac{1-F(\mu)}{f(\mu)}(1-q_0(\mu))dF(\mu) \leq \frac{1-F(1-\varepsilon_1)}{\varepsilon_1 f(1-\varepsilon_1)} \int_{\overline{\mu}}^{\overline{\mu}+\varepsilon_0} (1-\mu)(q_0(\mu) - q_0(\overline{\mu}))dF(\mu)$$

$$\implies \int_{1-\varepsilon_1}^{1} \frac{1-F(\mu)}{f(\mu)}(1-q_0(\mu))dF(\mu) \leq \int_{\overline{\mu}}^{\overline{\mu}+\varepsilon_0} \frac{1-F(\mu)}{f(\mu)}(q_0(\mu)-q_0(\overline{\mu}))dF(\mu)$$

The first implication follows from the choice of $\varepsilon_0, \varepsilon_1$. Thus any candidate allocation can be weakly improved by one for which there exists $\mu' \in [\overline{\mu}, 1]$ such that $q_0(\mu) = 0$ on $[\overline{\mu}, \mu')$ and $q_0(\mu) = 1$ on $[\mu', 1]$. Note that the latter implies item 1 in the statement of the Lemma in view of Equation 29.

In particular, by Equation 29, the improved allocation has $q_0(\mu) = 1$ for $\mu \in [\mu_0, \mu_1)$ and $q_0(\mu) = 0$ for $\mu \in [\mu_1, \mu')$. Next we claim that any such allocation for which μ_1 is strictly between μ_0 and μ' can be improved by one which satisfies item 2 in the statement of the Lemma. In other words, one for which μ_1 equals either μ_0 or μ'.

Assume for contradiction that the q is such that $\mu_0 < \mu_1 < \mu'$. We construct a new allocation q' such that on $q'(\mu) = q(\mu)$ for $\mu \in [\mu_0, \mu')^C$. On the interval $[\mu_0, \mu_1)$ we require $q_0'(\mu) = q_0(\mu) - \varepsilon_0$, on the interval $[\mu_1, \mu')$ we require $q_0'(\mu) = q_0(\mu) + \varepsilon_1$. Where

$$\varepsilon_1 = \varepsilon_0 \frac{\int_{\mu_0}^{\mu_1}(1-\mu)dF(\mu)}{\int_{\mu_1}^{\mu'}(1-\mu)dF(\mu)}$$

For small enough $\varepsilon_0 > 0$, the allocation q' is feasible with the same target mean as q. We argue that under the condition of the proposition, this q' achieves a weakly higher revenue. To see this, consider the difference in revenue between q and q':

$$\varepsilon_0 \int_{\mu_0}^{\mu_1} \left[1-\mu+\frac{1-F(\mu)}{f(\mu)}\right] dF(\mu) - \varepsilon_1 \int_{\mu_1}^{\mu'} \left[1-\mu+\frac{1-F(\mu)}{f(\mu)}\right] dF(\mu)$$

Similar to Proposition Proposition 1, when $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing the change in revenue is positive as

$$\frac{\int_{\mu_0}^{\mu_1} \frac{1-F(\mu)}{f(\mu)}dF(\mu)}{\int_{\mu_1}^{\mu'} \frac{1-F(\mu)}{f(\mu)}dF(\mu)} > \frac{\int_{\mu_0}^{\mu_1}(1-\mu)dF(\mu)}{\int_{\mu_1}^{\mu'}(1-\mu)dF(\mu)}.$$

□

B.4 Proof of Proposition 2

Proof of Proposition 2. It follows from Lemma 3, Lemma 4 and Equation 29 that a solution to the relaxed problem takes one of the following two forms.

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu < \mu_0 \\ \left(\frac{K(\mu)-p}{1-p}, 1\right) & \mu \in [\mu_0, \mu') \\ (1,1) & \mu > \mu' \end{cases} \tag{30}$$

or

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu < \mu_0 \\ \left(\frac{K(\mu)}{1-p}, 0\right) & \mu \in [\mu_0, \overline{\mu}) \\ (1,1) & \mu > \overline{\mu} \end{cases} \tag{31}$$

We complete the proof of Proposition 2 by identifying the slope in the middle regions. Consider the first case and let $K(\mu)$ be a feasible slope. By definition of μ_0 we have $K(\mu_0) - p \geq (1-\mu_0)K(\mu_0)$ implying $K(\mu_0) \geq \frac{p}{\mu_0}$. Therefore by monotonicity of K, for all $\mu \geq \mu_0$ we have $1 \geq K(\mu) \geq \frac{p}{\mu_0}$. We will show in fact that a solution to the relaxed problem in this case has $K(\mu) \in \left\{\frac{p}{\mu_0}, 1\right\}$ for all $\mu \in [\mu_0, \mu')$.

If $K(\mu)$ fails this property then we construct a new allocation that is determined by Equation 30 for slope function $\hat{K}$, where $\hat{K}(\mu) = K(\mu)$ for all $\mu \notin [\mu_0, \mu')$. Let $\hat{K}(\mu) = \frac{p}{\mu_0}$ for $\mu \in [\mu_0, \mu_0 + \varepsilon_0)$ and $\hat{K}(\mu) = 1$ for $\mu \in [\mu' - \varepsilon_1, \mu')$ and $\hat{K} = K$ otherwise. Choose $\varepsilon_0, \varepsilon_1 > 0$ such that $\mu_0 + \varepsilon_0 < \mu' - \varepsilon_1$ and the following holds

$$\int_{\mu'-\varepsilon_1}^{\mu'} \mu \left(1 - K(\mu)\right) dF(\mu) - \int_{\mu_0}^{\mu_0+\varepsilon_0} \mu \left(K(\mu) - \frac{p}{\mu_0}\right) dF(\mu) = 0 \tag{32}$$

By construction the allocation corresponding to $\hat{K}$ is feasible as it has the same mean as the one corresponding to K, this follows from Equation 32 and Equation 25 and the fact that $q_1(\mu) = \frac{K(\mu)}{1-p} + \textit{constant}$ over $[\mu_0, \mu')$ (by Equation 30). By assumption and monotonicity, there must also be values of ε_0, ε_1 such that $\hat{K} \neq K$.

The change in revenue is given by

$$\int_{\mu'-\varepsilon_1}^{\mu'} \left[\frac{\mu(m-p)}{(1-p)m} - \frac{1-F(\mu)}{f(\mu)}\right](1-K(\mu))\,dF(\mu) \\ - \int_{\mu_0}^{\mu_0+\varepsilon_0} \left[\frac{\mu(m-p)}{(1-p)m} - \frac{1-F(\mu)}{f(\mu)}\right]\left(K(\mu) - \frac{p}{\mu_0}\right)dF(\mu)$$

which by Equation 32 equals

$$= -\int_{\mu'-\varepsilon_1}^{\mu'} \frac{1-F(\mu)}{f(\mu)}(1-K(\mu))\,dF(\mu) + \int_{\mu_0}^{\mu_0+\varepsilon_0} \frac{1-F(\mu)}{f(\mu)}\left(K(\mu) - \frac{p}{\mu_0}\right)dF(\mu)$$

Note that $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non increasing, implies that $\frac{1-F(\mu)}{f(\mu)}$ is non increasing, thus the change in revenue is positive if

$$\frac{1-F(\mu_0+\varepsilon_0)}{\mu_0+\varepsilon_0}\left[\int_{\mu_0}^{\mu_0+\varepsilon_0}\left(K(\mu) - \frac{p}{\mu_0}\right)dF(\mu) - \int_{\mu'-\varepsilon_1}^{\mu'}(1-K(\mu))dF(\mu)\right] \geq 0$$

The inequality follows from the fact that $\frac{1-F(\mu_0+\varepsilon_0)}{\mu_0+\varepsilon_0}$ is positive and by Equation 32 which implies

$$\int_{\mu'-\varepsilon_1}^{\mu'}(1-K(\mu))dF(\mu) \leq \frac{\mu_0+\varepsilon_0}{\mu'-\varepsilon_1}\int_{\mu_0}^{\mu_0+\varepsilon_0}\left(K(\mu) - \frac{p}{\mu_0}\right)dF(\mu).$$

We conclude that a solution to the relaxed problem in the first case has $K(\mu) \in \left\{\frac{p}{\mu_0}, 1\right\}$ for all $\mu \in [\mu_0, \mu')$. We can now conclude the proof of the first case in the Proposition. It suffices to note that $K(\mu) = 0$ implies $q_1(\mu) = q_0(\mu) = 0$ and $K(\mu) = 1$ implies $q_1(\mu) = q_0(\mu) = 1$. Then for the interval $\mu \in [\mu_0, \mu')$ since $K(\mu) = \frac{p}{\mu_0}$ and $q(\mu) = \left(\frac{K(\mu)-p}{1-p}, 1\right)$ (by Equation 30) we obtain $q_1(\mu) = \frac{p}{1-p}\frac{1-\mu_0}{\mu_0}$.

The second case, Equation 31, is treated following similar lines. By definition of $\overline{\mu}$, we have $0 \leq K(\mu) \leq 1-p$ for $\mu \in [\mu_0, \overline{\mu})$. By arguments analogous to the first case we can show that a solution to the relaxed problem has $K(\mu) \in \{0, 1-p\}$ for $\mu \in [\mu_0, \overline{\mu})$. To conclude the proof note that for the interval $\mu \in [\mu_0, \overline{\mu})$ since $K(\mu) = 1-p$ and $q(\mu) = \left(\frac{K(\mu)}{1-p}, 0\right)$ (by Equation 31) we obtain $q_1(\mu) = 1$. $\square$

B.5 Optimal Mean

Proposition 2 characterizes the structure of the optimal mechanism for the relaxed problem. Using this structure, we can show that the optimal target mean for a given price p is $\mathbf{E}H_0^p$.

Lemma 5. *If* $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ *is non-increasing then the optimal mean in Equation 9 for a solution to the relaxed problem with price* p *is* $m = \mathbf{E}H_0^p$.

Proof. Consider for contradiction that q is of one of the forms identified in Proposition 2 but the corresponding target mean $m < \mathbf{E}H_0^p$.

Case I: Let q be of the first form

$$q(\mu) = \begin{cases} (0,0) & \text{if} \quad \mu \leq \mu_0 \\ \left(\frac{p}{1-p}\frac{1-\mu_0}{\mu_0}, 1\right) & \text{if} \quad \mu \in [\mu_0, \mu') \\ (1,1) & \quad\text{otherwise} \end{cases}$$

If $\mu_0 < \mu'$ then we can construct a profitable deviation by choosing $\delta^*, \delta > 0$ and by perturbing the allocation to $(0,0)$ on $[\mu_0, \mu_0 + \delta^*)$ and to $(1,1)$ on $[\mu' - \delta, \mu')$. We can choose δ, δ^* such that the new mean of the allocation is $m' \leq \mathbf{E}H_0^p$ and the following holds

$$\int_{\mu_0}^{\mu_0+\delta^*} \mu \frac{p}{1-p}\frac{1-\mu_0}{\mu_0}\, dF(\mu) = \int_{\mu'-\delta}^{\mu'} \mu \left(1 - \frac{p}{1-p}\frac{1-\mu_0}{\mu_0}\right) dF(\mu)$$

As $\frac{1-F(\mu)}{\mu f(\mu)}$ is non-increasing, this perturbation reduces the information rent and increases the gross surplus.

If $\mu_0 = \mu'$ then we can create a profitable perturbation by changing the allocation to $(1,0)$ on $[\mu', \mu' + \delta)$ for small enough $\delta > 0$.

Case II: Let q be of the second form

$$q(\mu) = \begin{cases} (0,0) & \text{if} \quad \mu \leq \mu_0 \\ (1,0) & \text{if} \quad \mu \in [\mu_0, \mu') \\ (1,1) & \quad\text{otherwise} \end{cases}$$

If $\mu_0 = \mu'$, we can repeat the previous argument.

If $\mu_0 < \mu'$ then we can construct a profitable deviation by choosing $\delta > 0$ and by perturbing the allocation to $(1,0)$ on $[\mu', \mu' + \delta^*)$. This is feasible for a small enough δ as $m < \mathbf{E}H_0^p \leq 1$ implies $\mu' < 1$. □

B.6 Uniform Prior– Theorem 1

So far, we have characterized the qualitative structure of the optimal relaxed mechanism under some distributional constraints. In this section, we demonstrate the power of our characterization by deriving the optimal primal mechanism for a uniform prior. Consider $F \sim \text{Unif}[0,1]$, this satisfies all conditions of Proposition 2, thus the optimal solution to the relaxed problem can be expressed as one of the mechanisms identified in the proposition. Moreover, by Lemma 5 it suffices to consider Equation 9 where $m = \mathbf{E}H_0^p$. First, we look at the mechanism of the form

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu < \mu_0 \\ (1,0) & \mu \in [\mu_0, \overline{\mu}) \\ (1,1) & \mu > \overline{\mu} \end{cases}$$

The mean constraint binding with no point mass at p implies that the relaxed problem can be written as

$$\max_{\mu_0, \overline{\mu}} \int_{\mu_0}^{1} (1-p)(2\mu - 1) d\mu - 2p \int_{\overline{\mu}}^{1} (1-\mu) d\mu$$

$s.t.$

$$0 \leq \mu_0 \leq \overline{\mu} \leq 1$$

$$\int_{\mu_0}^{1} \mu d\mu = \frac{\mathbf{E}H_0^p}{1 - \mathbf{E}H_0^p} \int_{\overline{\mu}}^{1} (1-\mu) d\mu$$

The objective can be rewritten as

$$\int_{\mu_0}^{1} (1-p)(2\mu - 1) d\mu - 2p \frac{1 - \mathbf{E}H_0^p}{\mathbf{E}H_0^p} \int_{\mu_0}^{1} \mu d\mu$$

Note that $\mu_0 = \mu^*(p) := \frac{1-p}{2\left(1 - \frac{p}{\mathbf{E}H_0^p}\right)}$ is the pointwise maximize of the objective. Let

$p_1 := \text{argmax} \left\{ p \in [0,1] \,\middle|\, \mu^*(p) \geq t(\mu^*(p)) \right\}$, where $t(\mu^*) = 1 - \sqrt{\frac{(1 - \mathbf{E}H_0^p)}{\mathbf{E}H_0^p} (1 - (\mu^*)^2)}$.

For $p \geq p_1$, the point-wise optimal mechanism is feasible for the relaxed problem. Moreover, the revenue is given by

$$\int_{\mu^*(p)}^{1} \left[2\mu \left(1 - \frac{p}{\mathbf{E}H_0^p}\right) - (1-p) \right] d\mu$$

The above is decreasing in p, the derivative of the revenue is

$$\int_{\mu^*(p)}^{1} 1 - \frac{2\mu}{p} \frac{1}{(1-\ln(p))^2} d\mu$$

This is negative as $\frac{1-3\ln(p)-p(1-\ln(p))}{-2p\ln(p)(1-\ln(p))^2} > 1$ for all $p \in [0,1]$.

For $p < p_1$, the point-wise optimal mechanism doesn't satisfy the interim participation constraint. The interim participation fails for $p < p_1$ as $\mu_1 < \mu^*(p)$. Note that the revenue decreases if $q_1(\mu)$ increases for $\mu < \mu^*(p)$. Thus, if $p < p_1$ then the optimal solution to the relaxed problem, restricted to mechanisms above, involves thresholds $\mu_0 = \overline{\mu}$. By the mean constraint, the threshold value μ_0 is a root of the following quadratic equation

$$(1-\overline{\mu})^2 = \frac{1-\mathbf{E}H_0^p}{\mathbf{E}H_0^p}(1-\overline{\mu}^2)$$

This has a single interior solution, $\mu_0 = \overline{\mu} = 2\mathbf{E}H_0^p - 1$, the revenue is thus given by

$$\int_{2\mathbf{E}H_0^p-1}^{1} \left[2\mu\left(1-\frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right] d\mu$$

The above is negative if $p_1 > p \geq p_0 := \operatorname{argmax}\left\{p \in [0,1] \;\middle|\; \frac{1+p}{2} \geq p(1-\ln p)\right\}$, thus the optimal thresholds are such that is if $p \in [p_0, p_1]$ then $\mu_0 = \overline{\mu} = 2\mathbf{E}H_0^p - 1$ and if $p < p_0$ then $\mu_0 = 1$. Thus, to find the optimal mechanism in the first class of mechanisms from the Proposition 2, we will maximize the following objective with respect to $p \in [p_0, p_1]$.

$$R(p) := \int_{2\mathbf{E}H_0^p-1}^{1} \left[2\mu\left(1-\frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right] d\mu = 2\left[1-\mathbf{E}H_0^p\right]\left[2\mathbf{E}H_0^p - (1+p)\right]$$

This function is concave in the domain and achieves an interior maximum. Unfortunately, the first order condition is a transcendental equation, so we rely on numerical methods to calculate the optimum. The optimal revenue for the relaxed problem among this class of mechanisms is $\simeq 0.0646$, which is achieved by $p \simeq 0.4364$, and $\mu_0 = \overline{\mu} \simeq 0.596$.

Now we will describe the optimal mechanism for the second class of mechanisms identified in Proposition 2;

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu < \mu_0 \\ \left(\frac{p(1-\mu_0)}{(1-p)\mu_0}, 1\right) & \mu \in [\mu_0, \overline{\mu}) \\ (1,1) & \mu > \overline{\mu} \end{cases}$$

We only need to optimize over these mechanisms for price $p < p_1$, as for other prices the mechanism discussed above maximizes the revenue point-wise. We solve the following problem

$$\max_{\mu_0, \overline{\mu}} \int_{\mu_0}^{\overline{\mu}} (1-p)\frac{p(1-\mu_0)}{(1-p)\mu_0}(2\mu - 1)d\mu + \int_{\overline{\mu}}^{1} (1-p)(2\mu-1)d\mu - 2p\int_{\mu_0}^{1}(1-\mu)d\mu$$

s.t.

$$p \leq \mu_0 \leq \overline{\mu} \leq 1$$

$$\int_{\mu_0}^{\overline{\mu}} \mu \frac{p(1-\mu_0)}{(1-p)\mu_0} d\mu + \int_{\mu_0}^{1} \mu d\mu = \frac{\mathbf{E}H_0^p}{1-\mathbf{E}H_0^p}\int_{\mu_0}^{1}(1-\mu)d\mu$$

The objective can be rewritten as

$$\int_{\mu_0}^{\overline{\mu}} \frac{p(1-\mu_0)}{(1-p)\mu_0}\left[2\mu\left(1-\frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right]d\mu + \int_{\overline{\mu}}^{1}\left[2\mu\left(1-\frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right]d\mu$$

From the obedience we can express $\overline{\mu} = h(\mu_0) := \sqrt{\mu_0^2 + \frac{(1-p)\mu_0(1-\mu_0)}{(\mu_0-p)(1-\mathbf{E}H_0^p)}(1+\mu_0 - 2\mathbf{E}H_0^p)}$

For feasibility we require $p \leq \mu_0 \leq \overline{\mu}$, thus feasibility can be restated as $\max\{p, 2\mathbf{E}H_0^p - 1\} \leq \mu_0$. The optimization problem can be restated

$$\int_{\mu_0}^{h(\mu_0)} \frac{p(1-\mu_0)}{(1-p)\mu_0}\left[2\mu\left(1-\frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right]d\mu + \int_{h(\mu_0)}^{1}\left[2\mu\left(1-\frac{p}{\mathbf{E}H_0^p}\right) - (1-p)\right]d\mu$$

s.t.

$$\max\{p, 2\mathbf{E}H_0^p - 1\} \leq \mu_0 \leq 1$$

This is a well-defined two-variable unconstrained optimization on a compact set. In particular, we can numerically derive the relevant features of the mechanism at the optimum. The optimal revenue for the relaxed problem among this class of mechanisms is $\simeq 0.0651$, which is achieved by $p^* \simeq 0.417$, $\mu_0 \simeq 0.578$, and $\overline{\mu} \simeq 0.629$. Thus optimal relaxed mechanism with revenue-maximizing price p^* is given by

$$(q_1(\mu), q_0(\mu)) = \begin{cases} (0,0) & \mu < 0.578 \\ (0.522, 1) & \mu \in [0.578, 0.629) \\ (1,1) & \mu > 0.629 \end{cases}$$

The CDF for the receiver's second-order beliefs for the above mechanism is given by

$$\operatorname{marg}_{\Delta(\theta)} \beta(\mu \mid p = 0.417) = \begin{cases} 0 & \mu < 0.417 \\ w^{-1} \int_{0.578}^{\frac{\mu}{\mu+0.522(1-\mu)}} (0.522s + (1-s))ds & \mu \in [0.417, 0.47) \\ w^{-1} \int_{0.578}^{0.629} (0.522s + (1-s))ds & \mu \in [0.47, 0.629) \\ \frac{\left[\int_{0.578}^{0.629} (0.522s + (1-s))ds + \int_{0.629}^{\mu} (s + (1-s))ds\right]}{w} & \mu \geq 0.629 \end{cases}$$

Where $w = \int_{0.578}^{0.629} 0.522s ds + \int_{0.629}^{1} ds + \int_{0.578}^{0.629} (1-s)\, ds$. As the mechanism solves the relaxed problem, $\operatorname{marg}_{\Delta(\theta)} \beta(\cdot \mid p = 0.417)$ has the same expectation as $H_0^{0.417}$. We claim that $H_0^{0.417} \succeq_{\text{mps}} \operatorname{marg}_{\Delta(\theta)} \beta(\cdot \mid p = 0.417)$, this follows from the fact that the distributions have the same support and that $\operatorname{marg}_{\Delta(\theta)} \beta(\cdot \mid p = 0.417)$ crosses $H_0^{0.417}$ exactly once from below.

B.7 Deterring Double Deviations

If the platform can leverage the risk-neutrality of the seller and enforce random prices with expected value p, then a scheme to deter double deviations is feasible. It works as follows. Consider any type μ which joins the platform with positive probability in the optimal mechanism. Let ν be the expected posterior obtained by μ conditional on joining the platform. We have that $\nu \geq p$. Now when any other type $\underline{\mu}$ misreports as μ, they will obtain a random posterior $\tilde{v}(\underline{\mu})$ with some expectation $\mathbf{E}\tilde{v}(\underline{\mu})$. Let $\underline{\mu} \leq \mu$ be the type such that the expected posterior that $\underline{\mu}$ would obtain from mis-reporting as μ equals p, i.e. $\mathbf{E}\tilde{v}(\underline{\mu}) = p$. The continuity and monotonicity of Bayesian updating guarantees that $\underline{\mu}$ exists and is unique.

When μ joins the platform require the firm to charge price equal to the realization of $\tilde{v}(\underline{\mu})$. Note that this price is in a one-to-one correspondence with the realized posterior of type μ as well as the posterior that would be obtained if any other type were to misreport μ. Note that this random price has expected value p so it leaves all payoffs, revenue, and incentive constraints unchanged.

Now consider double deviations. Consider any type μ' which misreports as μ and must then decide whether to accept the offered price. By the monotonicity of Bayesian updating if $\mu' \geq \mu$ the realized price is below $\tilde{v}(\mu')$ with probability 1. Therefore, types higher than μ find no double deviation profitable. On the other hand if $\mu' < \mu$ then after misreporting as μ, the type μ' finds every price realization strictly below $\tilde{v}(\mu')$, again by the strict monotonicity of Bayesian updating. Such a type therefore rejects every price offered and obtains a payoff from zero from the mis-report. Since the participation constraint already requires that the payoff from truth-telling is weakly greater than zero, these double deviations are never profitable.

C Credit Rating

C.1 Obedience

The given interest rate r, the iso-elastic risk profile H_0^r is such that Equation 12 with $\operatorname{marg}_{\Delta\theta} \beta(\cdot \mid r) = H_0^r$ hold with equality for all $\nu \geq \hat{\mu}(r)$. In particular, we have the following condition for all $r' \in [r, \hat{r}(1)]$

$$\mathbf{E}_{H_0^r}\left[r' - \nu(1+r')\left(1+\frac{\overline{R}}{R_1}\right)\middle|\nu \geq \hat{\mu}(r')\right] = \left(\hat{r}(1) - (1+\hat{r}(1))\left(1-\frac{\overline{R}}{R_1}\right)\right)\lim_{\mu'\uparrow 1} H_0^r(\mu')$$

Equivalently

$$\int_{\mu}^{1}\left(\hat{r}(\mu) - \nu(1+\hat{r}(\mu))\left(1-\frac{\overline{R}}{R_1}\right)\right)dH_0^r(\nu) \quad = \quad (\overline{R} - R_0 - 1)\lim_{\mu'\uparrow 1} H_0^r(\mu') \quad (33)$$

Adding and subtracting $\int_\mu^1 \left(\hat{r}(\mu) - \mu(1+\hat{r}(\mu))\left(1 - \frac{\overline{R}}{R_1}\right)\right) dH_0^r(\nu)$ to the left-hand side yields the following equality

$$\int_\mu^1 (\mu - \nu)(1+\hat{r}(\mu))\left(1 - \frac{\overline{R}}{R_1}\right) dH_0^r(\nu) + \int_\mu^1 \left(\hat{r}(\mu) - \mu(1+\hat{r}(\mu))\left(1 - \frac{\overline{R}}{R_1}\right)\right) dH_0^r(\nu)$$
$$= \left(\overline{R} - R_0 - 1\right)\left(1 - \lim_{\mu' \uparrow 1} H_0^r(\mu')\right) \quad (34)$$

Equivalently, we have the following ordinary differential equation

$$(1+\hat{r}(\mu))\left(1 - \frac{\overline{R}}{R_1}\right) \int_\mu^1 (\mu - \nu) dH_0^r(\nu) + \left(\hat{r}(\mu) - \mu(1+\hat{r}(\mu))\left(1 - \frac{\overline{R}}{R_1}\right)\right) \frac{\partial}{\partial \mu} \int_\mu^1 (\mu - \nu) dH_0^r(\nu)$$
$$= \left(\overline{R} - R_0 - 1\right)\left(1 - \lim_{\mu' \uparrow 1} H_0^r(\mu')\right)$$

Dividing both sides by $\hat{r}(\mu) - \mu(1+\hat{r}(\mu))\left(1 - \frac{\overline{R}}{R_1}\right)$ and plugging in $1 + \hat{r}(\mu) = \frac{\overline{R} - R_0}{1 - \mu\left(1 - \frac{\overline{R}}{R_1}\right)}$ we get that

$$\hat{r}(\mu) - \mu(1+\hat{r}(\mu))\left(1 - \frac{\overline{R}}{R_1}\right) = \overline{R} - R_0 - 1$$

and

$$g(\mu) := \frac{(1+\hat{r}(\mu))\left(1 - \frac{\overline{R}}{R_1}\right)}{\hat{r}(\mu) - \mu(1+\hat{r}(\mu))\left(1 - \frac{\overline{R}}{R_1}\right)} = \frac{\overline{R} - R_0}{\overline{R} - R_0 - 1} \frac{1 - \frac{\overline{R}}{R_1}}{1 - \mu\left(1 - \frac{\overline{R}}{R_1}\right)}$$

Thus we recover Equation 17

$$\frac{d}{d\mu}\psi(\mu) + g(\mu)\psi(\mu) = 1 - \lim_{\mu' \uparrow 1} H_0^r(\mu')$$

Using the integration factor $I(\mu) = \exp\left(-\int_\mu^1 g(\mu')d\mu'\right)$ we get that

$$\frac{d}{d\mu}\left(\psi(\mu) \times I(\mu)\right) = I(\mu)(1 - \lim_{\mu' \uparrow 1} H_0^r(\mu'))$$

The solution to this differential equation is given by

$$\psi(\mu) = -(1 - \lim_{\mu' \uparrow 1} H_0^r(\mu')) \times \int_\mu^1 \frac{I(\nu)}{I(\mu)} d\nu \tag{35}$$

Differentiating and plugging in the initial condition $\frac{d}{d\mu}\psi(\mu)\Big|_{\mu=\hat{\mu}(r)} = 1$ we get that

$$H_0^r(\nu) = \begin{cases} 0 & \nu \leq \hat{\mu}(r) \\ 1 - \frac{J(\nu)}{J(\hat{\mu}(r))} & \hat{\mu}(r) < \nu < 1 \\ 1 & \nu = 1 \end{cases}$$

Where $J(\nu) = 1 + g(\nu) \int_\nu^1 \frac{I(\mu')}{I(\nu)} d\mu'$.

To show that $H_0^r(\nu)$ is a well-defined cumulative distribution function, we need to verify that it is non-decreasing in ν. By definition $H_0^r(\nu) = 1 - \frac{J(\nu)}{J(\hat{\mu}(r))}$ for $\hat{\mu}(r) < \nu < 1$. Differentiating $H_0^r(\nu)$ with respect to ν we get the following

$$\frac{\partial}{\partial \nu} H_0^r(\nu) = \frac{-1}{J(\hat{\mu}(r))} \frac{d}{d\nu} J(\nu)$$

The above is positive as $J(\hat{\mu}(r)) > 0$ and as

$$\begin{aligned} \frac{d}{d\nu} J(\nu) &= \left(\frac{d}{d\nu} g(\nu)\right) \int_\nu^1 \frac{I(\mu')}{I(\nu)} d\mu' - g^2(\nu) \int_\nu^1 \frac{I(\mu')}{I(\nu)} d\mu' - g(\nu) \\ &= \left(\frac{d}{d\nu} g(\nu) - g^2(\nu)\right) \int_\nu^1 \frac{I(\mu')}{I(\nu)} d\mu' - g(\nu) \\ &= \frac{\overline{R} - R_0}{\overline{R} - R_0 - 1} \left(1 - \frac{\overline{R}}{R_1}\right)^2 \left(\frac{1}{1 - \nu\left(1 - \frac{\overline{R}}{R_1}\right)}\right)^2 \left(\frac{-1}{\overline{R} - R_0 - 1}\right) \int_\nu^1 \frac{I(\mu')}{I(\nu)} d\mu' - g(\nu) \\ &= -g^2(\nu) \frac{1}{\overline{R} - R_0} \int_\nu^1 \frac{I(\mu')}{I(\nu)} d\mu' - g(\nu) \\ &= -g(\nu) \frac{J(\nu) + \overline{R} - R_0 - 1}{\overline{R} - R_0} < 0 \end{aligned}$$

The last inequality holds as $\overline{R} - R_0 - 1 > 0$. From above, we get that $J(\nu)$ is decreasing in ν and hence H_0^r is non-decreasing.

By Equation 35 the mean of H_0^r is given by

$$
\begin{aligned}
\mathbf{E}H_0^r &= \hat{\mu}(r) + (1 - \lim_{\mu' \uparrow 1} H_0^r(\mu')) \times \int_{\hat{\mu}(r)}^1 \frac{I(\nu)}{I(\hat{\mu}(r))} d\nu \\
&= \hat{\mu}(r) + \frac{1}{J(\hat{\mu}(r))} \int_{\hat{\mu}(r)}^1 \frac{I(\nu)}{I(\hat{\mu}(r))} d\nu \\
\implies \frac{d}{dr}\mathbf{E}H_0^r &= \left[1 + \frac{d}{d\hat{\mu}(r)} \left(\frac{1}{J(\hat{\mu}(r))} \int_{\hat{\mu}(r)}^1 \frac{I(\nu)}{I(\hat{\mu}(r))} d\nu \right) \right] \frac{d}{dr}\hat{\mu}(r) \\
&= \int_{\hat{\mu}(r)}^1 \frac{I(\nu)}{I(\hat{\mu}(r))} d\nu \times \frac{d}{d\hat{\mu}(r)} \frac{1}{J(\hat{\mu}(r))} \times \frac{d}{dr}\hat{\mu}(r)
\end{aligned}
$$

As $\hat{\mu}(r)$ is increasing in r and as $J(\mu)$ is decreasing in μ, we get that $\mathbf{E}H_0^r$ is increasing in r.

In addition to H_0^r, we have a whole family of distributions H_x^r indexed by the size of the mass point at $\hat{\mu}(r)$, which are isoelastic on their support, and the lender optimally sets the rate r.

H_x^r is given by the following

$$
H_x^r(\nu) = \begin{cases} 0 & \nu < \hat{\mu}(r) \\ x & \hat{\mu}(r) \leq \nu \leq \widetilde{\mu}(x) \\ x + (1-x) H_0^{\hat{r}(\widetilde{\mu}(x))}(\nu) & \widetilde{\mu}(x) < \nu < 1 \\ 1 & \nu = 1 \end{cases}
$$

For every value of x there exists a type $\widetilde{\mu}(x) \in [\hat{\mu}(r), 1]$ such that the support of H_x^r is $\{\hat{\mu}(r)\} \cup [\widetilde{\mu}(x), 1]$ and H_x^r is iso elastic on its support.

Let $x_r = \frac{1}{r}\left(R_1 - R_0 - (1+r)\frac{\overline{R}}{R_1}\right)$. Then $\widetilde{\mu}(x)$ is the solution to the following equality when $x \leq x_r$ and $\widetilde{\mu}(x) = 1$ when $x > x_r$

$$
r - (1+r)\left(x\hat{\mu}(r) + (1-x)\mathbf{E}H_0^{\hat{r}(\widetilde{\mu}(x))} \right) \left(1 - \frac{\overline{R}}{R_1} \right) = (1-x)(R_1 - R_0 - 1)\left(1 - \lim_{\mu' \uparrow 1} H_0^{\hat{r}(\widetilde{\mu}(x))}(\mu') \right) \tag{36}
$$

We have shown that $J(\nu)$ is decreasing in ν. Thus, the left-hand side above is decreasing in $\widetilde{\mu}(x)$ as $\mathbf{E}H_0^{r'}$ increases in r', and the right-hand side is increasing in $\widetilde{\mu}(x)$ as $\lim_{\mu' \uparrow 1} H_0^{r'}(\mu') = 1 - \frac{1}{J(\hat{\mu}(r'))}$. This implies that $\widetilde{\mu}(x)$ is well defined.

Before we present the main result of this section, that any distribution satisfying Equation 14 and Equation 12 is a mean-preserving contraction of an iso elastic distribution identified above, we will show that any such distribution must have a mean $m \leq \mathbf{E}H_0^r$.

Lemma 6. *Any distribution G that satisfies Equation 14 and Equation 12 is such that $\mathbf{E}G \leq \mathbf{E}H_0^r$.*

Proof. Assume for contradiction that G satisfies Equation 12, Equation 14 and has a mean $m > \mathbf{E}H_0^r$. By Equation 12 we must have the following

$$(\overline{R} - R_0 - 1)(1 - \lim_{\mu' \uparrow 1} G(\mu')) \leq r - m(1+r)\left(1 - \frac{\overline{R}}{R_1}\right)$$

By assumption $m > \mathbf{E}H_0^r$ thus we get

$$(\overline{R} - R_0 - 1)(1 - \lim_{\mu' \uparrow 1} G(\mu')) < r - (1+r)\left(1 - \frac{\overline{R}}{R_1}\right)\mathbf{E}H_0^r$$

Combining the above with Equation 33 for H_0^r we get that

$$\lim_{\mu' \uparrow 1} G(\mu') > \lim_{\mu' \uparrow 1} H_0^r(\mu')$$

As $\mathbf{E}G > \mathbf{E}H_0^r$ the CDF G can not be everywhere above H_0^r. By continuity of H_0^r and right continuity of G we get that there exists some $\hat{\mu}(r) < \mu_0 < 1$ such that $\lim_{\mu' \uparrow \mu_0} G(\mu') \leq H_0^r(\mu_0) \leq G(\mu_0)$ and $H_0^r(\mu) \leq G(\mu)$ for all $\mu > \mu_0$. Evaluating the lender's payoff from setting a rate $\hat{r}(\mu_0)$ against the distribution G we get

$$\hat{r}(\mu_0)(1 - \lim_{\mu' \uparrow \mu_0} G(\mu')) - (1+\hat{r}(\mu_0))\left(1 - \frac{\overline{R}}{R_1}\right)\int_{[\mu_0,1]} \nu dG(\nu)$$
$$\geq$$
$$\hat{r}(\mu_0)(1 - \lim_{\mu' \uparrow \mu_0} G(\mu')) - (1+\hat{r}(\mu_0))\left(1 - \frac{\overline{R}}{R_1}\right)\int_{[\mu_0,1]} \nu dG(\nu) - (\overline{R} - R_0 - 1)(H_0^r(\mu_0) - G(\mu_0))$$

As H_0^r is (weakly) below G for all $\mu > \mu_0$, we get that the above expression is bounded below by the following

$$\hat{r}(\mu_0)(1 - H_0^r(\mu_0)) - (1+\hat{r}(\mu_0))\left(1 - \frac{\overline{R}}{R_1}\right)\int_{\mu_0}^{1} \nu dH_0^r(\nu)$$

By Equation 33 the above equals

$$r - (1+r)\left(1 - \frac{\overline{R}}{R_1}\right)\mathbf{E}H_0^r$$

Thus, if $m > \mathbf{E}H_0^r$, setting a rate $\hat{r}(\mu_0)$ yields greater payoff to the lender than setting a rate r, hence contradicting Equation 12. □

Proposition 4. *Given $r \in [\hat{r}(0), \hat{r}(1)]$. If G satisfies Equation 12 and Equation 14 with mean $m \in [\hat{\mu}(r), \mathbf{E}H_0^r]$ then G is a mean preserving contraction of H_x^r for some $x \in [0,1]$.*

Proof. *Case I:* The mean $\mathbf{E}G = m$ is such that

$$r - m(1+r))\left(1 - \frac{\overline{R}}{R_1}\right) \geq (\overline{R} - R_0 - 1)\left(\frac{m - \hat{\mu}(r)}{1 - \hat{\mu}(r)}\right)$$

A mean-preserving spread of G can be generated by type-by-type garbling all types in $(\hat{\mu}(r), 1)$ to posterior in $\{\hat{\mu}(r), 1\}$. By the above inequality, both Equation 12 and Equation 14 are satisfied by the resulting distribution.

Case II: The mean $\mathbf{E}G = m$ is such that

$$r - m(1+r))\left(1 - \frac{\overline{R}}{R_1}\right) < (\overline{R} - R_0 - 1)\left(\frac{m - \hat{\mu}(r)}{1 - \hat{\mu}(r)}\right)$$

By Equation 36 we get that $\widetilde{\mu}(x) < 1$ and $\mathbf{E}H_x^r = m$.

Consider $\mu \geq \widetilde{\mu}(x)$ then by Equation 33 for H_0^r and Equation 12 for G we get that following

$$\int_{\mu}^{1}\left(\hat{r}(\mu) - \nu(1+\hat{r}(\mu))\left(1 - \frac{\overline{R}}{R_1}\right)\right) dH_0^r(\nu) \geq \int_{[\mu,1]}\left(\hat{r}(\mu) - \nu(1+\hat{r}(\mu))\left(1 - \frac{\overline{R}}{R_1}\right)\right) dG(\nu)$$

The integral on the right-hand side accounts for G having a mass point at μ. Formally, it is the limit of integrals over the interval $(\mu - \varepsilon, 1]$ as ε tends to zero. By a similar manipulation as before, we can express this inequality in terms of a differential equation

$$g(\mu)\int_{\mu}^{1}(\nu - \mu)(dG(\nu) - dH_x^r(\nu)) + D_L\left(\int_{\mu}^{1}(\nu - \mu)(dG(\nu) - dH_x^r(\nu))\right) \geq 0$$

Here D_L is the left derivative which exists as $\int_\mu^1 (\nu-\mu)(dG(\nu)-dH_x^r(\nu)) = \int_0^1 \max\{\nu-\mu,0\}\ (dG(\nu)-dH_x^r(\nu))$ is convex in μ. The left derivative is necessary to accommodate any mass point that G might have at μ. Using the integration factor $I(\mu)$ we obtain the following

$$D_L\left(I(\mu)\int_\mu^1 (\nu-\mu)(dG(\nu)-dH_x^r(\nu))\right) \geq 0$$

As $I(\mu)\int_\mu^1 (\nu-\mu)(dG(\nu)-dH_x^r(\nu))$ is continous in μ and as the left derivative is everywhere above 0 for $\mu\in[\widetilde{\mu}(x),1]$, we get that $I(\mu)\int_\mu^1 (\nu-\mu)(dG(\nu)-dH_x^r(\nu))$ is increasing in μ. Plugging in $\mu=1$ yields us $I(1)\int_1^1 (\nu-\mu)(dG(\nu)-dH_x^r(\nu)) = 0$. As $I(\mu)>0$ we get that

$$\int_\mu^1 (\nu-\mu)(dG(\nu)-dH_x^r(\nu)) \leq 0$$

or equivalently for all $\mu\in[\widetilde{\mu}(x),1]$

$$\int_0^1 \max\{\nu-\mu,0\}\ dG(\nu) \leq \int_0^1 \max\{\nu-\mu,0\}\ dH_x^r(\nu) \tag{37}$$

Now consider $\mu\in[\hat{\mu}(r),H_x^r]$, by convexity of $\int_0^1 \max\{\nu-\mu,0\}\ dG(\nu)$ we obtain that $\int_0^1 \max\{\nu-\mu,0\}\ dG(\nu)$is bounded above by the following

$$\frac{\widetilde{\mu}(x)-\mu}{\widetilde{\mu}(x)-\hat{\mu}(r)}\int_0^1 \max\{\nu-\hat{\mu}(r),0\}\ dG(\nu) + \frac{\mu-\hat{\mu}(r)}{\widetilde{\mu}(x)-\hat{\mu}(r)}\int_0^1 \max\{\nu-\widetilde{\mu}(x),0\}\ dG(\nu)$$

As $\mathbf{E}G=\mathbf{E}H_x^r$ and both G and H_x^r satisfy Equation 14 we get that

$$\int_0^1 \max\{\nu-\hat{\mu}(r),0\}\ dG(\nu) = \int_0^1 \max\{\nu-\hat{\mu}(r),0\}\ dH_x^r(\nu)$$

Combining the above with Equation 37 we get that

$$\begin{aligned}&\frac{\widetilde{\mu}(x)-\mu}{\widetilde{\mu}(x)-\hat{\mu}(r)}\int_0^1 \max\{\nu-\hat{\mu}(r),0\}\ dG(\nu) + \frac{\mu-\hat{\mu}(r)}{\widetilde{\mu}(x)-\hat{\mu}(r)}\int_0^1 \max\{\nu-\widetilde{\mu}(x),0\}\ dG(\nu)\\ \leq\ &\frac{\widetilde{\mu}(x)-\mu}{\widetilde{\mu}(x)-\hat{\mu}(r)}\int_0^1 \max\{\nu-\hat{\mu}(r),0\}\ H_x^r(\nu) + \frac{\mu-\hat{\mu}(r)}{\widetilde{\mu}(x)-\hat{\mu}(r)}\int_0^1 \max\{\nu-\widetilde{\mu}(x),0\}\ dH_x^r(\nu)\end{aligned}$$

By definition of H_x^r the expression $\int_0^1 \max\{\nu-\mu,0\}\, dH_x^r(\nu)$ is linear in μ thus we obtain that for all $\mu \in [\hat{\mu}(r), \widetilde{\mu}(x)]$ the following holds

$$\int_0^1 \max\{\nu-\mu,0\}\, dG(\nu) \le \int_0^1 \max\{\nu-\mu,0\}\, dH_x^r(\nu)$$

The above together with Equation 37 implies that G is a mean-preserving contraction of H_x^r. □

C.1.1 Shape of H_x^r

Lemma 7. *If $R_1 - R_0 - 1 < 1$ then H_0^r is concave.*

Proof. Differentiating H_0^r twice with respect to ν gives us the following

$$\begin{aligned}\frac{\partial^2}{\partial^2\nu}H_0^r(\nu) &= \frac{1}{J(\hat{\mu}(r))}\frac{d}{d\nu}\left(g(\nu)\left(1+\frac{J(\nu)-1}{\overline{R}-R_0}\right)\right)\\ &= \frac{1}{J(\hat{\mu}(r))}\left(\frac{d}{d\nu}g(\nu)\left(1+\frac{J(\nu)-1}{\overline{R}-R_0}\right)+g(\nu)\frac{d}{d\nu}\frac{J(\nu)-1}{\overline{R}-R_0}\right)\\ &= \frac{1}{J(\hat{\mu}(r))}\left(\frac{d}{d\nu}g(\nu)\left(1+\frac{J(\nu)-1}{\overline{R}-R_0}\right)-\frac{g^2(\nu)}{\overline{R}-R_0}\left(1+\frac{J(\nu)}{\overline{R}-R_0}\right)\right)\\ &= \frac{1}{J(\hat{\mu}(r))}\left(1+\frac{J(\nu)}{\overline{R}-R_0}\right)\left(\frac{d}{d\nu}g(\nu)-\frac{g^2(\nu)}{\overline{R}-R_0}\right)-\frac{1}{J(\hat{\mu}(r))(\overline{R}-R_0)}\frac{d}{d\nu}g(\nu)\\ &< \frac{1}{J(\hat{\mu}(r))}\left(1+\frac{J(\nu)}{\overline{R}-R_0}\right)g^2(\nu)\frac{\overline{R}-R_0-1}{\overline{R}-R_0}\left(1-\frac{1}{\overline{R}-R_0-1}\right)<0\end{aligned}$$

The first strict inequality above follows as $\frac{d}{d\nu}g(\nu) > 0$. The last inequality follows as $\overline{R}-R_0 > 1 > \overline{R}-R_0-1$. Thus H_0^r is concave and increasing on $[\hat{\mu}(r), 1]$. □

C.2 Relaxed Problem

Any distribution of risk $\text{marg}_{\Delta\theta}\, \beta(\cdot \mid r)$ which is a mean-preserving contraction of H_x^r for some $x \in [\hat{\mu}(r), 1]$ can be transformed (via information disclosure) into a risk profile that satisfies Equation 12 and Equation 14. Similar to our platform example, we obtain Equation 18 and Equation 19 as necessary conditions for $\text{marg}_{\Delta\theta}\, \beta(\cdot \mid r)$ to be mean preserving contraction of H_x^r for some $x \in [\hat{\mu}(r), 1]$

$$\mathbf{E}\, \text{marg}_{\Delta\theta}\, \beta(\cdot \mid r) = m$$

and

$$\operatorname{marg}_{\Delta\theta} \beta(\hat{\mu}(r) \mid r) \leq x$$

Where $m = \mathbf{E}H_x^r \in [\hat{\mu}(r), \mathbf{E}H_0^r]$.

Recall that $K(\mu) = u_1(r)q_1(r) - u_0(\mu)q_0(\mu)$ is the slope of the indirect utility. We may substitute into the objective in Equation 20 to obtain

$$\begin{aligned}\Pi = &\int_0^1 \mu u_1(r)q_1(\mu)dF(\mu) + \int_0^1 (1-\mu)u_0(r)q_0(\mu)dF(\mu) - R_0\int_0^1 \overline{q}(\mu)dF(\mu) \\ &- \int_0^1 K(\mu)\left(\frac{1-F(\mu)}{f(\mu)}\right)dF(\mu) \\ = &\int_0^1 \mu\left(u_1(r) - R_0\right)q_1(\mu)dF(\mu) + \int_0^1 (1-\mu)\left(u_0(r) - R_0\right)q_0(\mu)dF(\mu) \\ &- \int_0^1 K(\mu)\left(\frac{1-F(\mu)}{f(\mu)}\right)dF(\mu)\end{aligned}$$

For any incentive-compatible and obedient mechanism q, there is a corresponding monotone slope function $K(\mu)$ taking values in $[0,1]$ and a target mean m defined by

$$\int_0^1 \mu q_1(\mu)dF(\mu) = \frac{m}{1-m}\int_0^1 (1-\mu)q_0(\mu)dF(\mu) \tag{38}$$

or

$$\int_0^1 \left[(1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right]q_1(\mu)dF(\mu) = \int_0^1 (1-\mu)K(\mu)dF(\mu)$$

Substituting Equation 38 into the objective and re-arranging we arrive at:

$$\Pi = \int_0^1 \mu\left(u_1(r) - R_0 + \left(u_0(r) - R_0\right)\frac{1-m}{m}\right)q_1(\mu)dF(\mu) - \int_0^1 \frac{1-F(\mu)}{f(\mu)}K(\mu)dF(\mu) \tag{39}$$

Like before, we have organized the objective function in a way that isolates the gains from increasing $q_1(\mu)$ from the costs of increasing the slope $K(\mu)$. In the background we can adjust $q_0(\mu)$ to maintain a given slope $K(\mu)$ and target mean m provided $q_1(\mu) \in [0,1]$ satisfies

$$\frac{K(\mu)}{u_1(r)} \leq q_1(\mu) \leq \min\left\{1,\ \frac{K(\mu) + u_0(r)}{u_1(r)}\right\} \tag{40}$$

The first inequality ensures $q_1(\mu), q_0(\mu) \geq 0$ and the second ensures $q_1(\mu), q_0(\mu) \leq 1$. We can also express the participation constraint Equation 14 in terms of $K(\mu)$ as following whenever $u_1(r) - \alpha_r(\mu)u_0(r) > 0$

$$q_1(\mu) \leq \frac{K(\mu)}{u_1(r) - \alpha_r(\mu)u_0(r)} \tag{41}$$

Where $\alpha_r(\mu) = \frac{\mu(1-\hat{\mu}(r))}{(1-\mu)\hat{\mu}(r)}$.

We will say that a slope function $K(\mu)$ is *feasible* with respect to a target mean m if there exists an allocation $q_1(\mu)$ which satisfies Equation 38, Equation 40, and Equation 41.

C.3 Optimal Allocation for Fixed K

We can approach the problem by first finding the optimal q_1 for a given target mean m and feasible $K(\mu)$, and then optimizing the latter.

Note that $\frac{K(\mu)}{u_1(r)-\alpha_r(\mu)u_0(r)}$ crosses $\min\left\{1, \frac{K(\mu)+u_0(r)}{u_1(r)}\right\}$ once and from above and at a point μ_0 for which $u_1(r) - \alpha_r(\mu_0)u_0(r) > 0$.

Moreover, on $[0, \mu_0]$ we have $\frac{K(\mu)}{u_1(r)-\alpha_r(\mu)u_0(r)} \geq \frac{K(\mu)}{u_1(r)}$.

Define the following class of allocations for some $\mu^K \in [0, 1]$

$$q_1^K(\mu) := \begin{cases} \frac{K(\mu)}{u_1(r)} & \mu < \mu^K \\ \frac{K(\mu)}{u_1(r)-\alpha_r(\mu)u_0(r)} & \mu_0 > \mu \geq \mu^K \\ \min\left\{1, \frac{K(\mu)+u_0(r)}{u_1(r)}\right\} & \overline{\mu} > \mu \geq \mu_0 \\ \frac{K(\mu)}{u_1(r)} & \mu \geq \overline{\mu} \end{cases}$$

Where $\mu_0 < \frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(r)} \leq \overline{\mu}$.

Lemma 8. *For given $m \in [\hat{\mu}(r), \mathbf{E}H_0^r]$ and feasible $K(\mu)$, the optimal allocation is $q_1 = q_1^K$ for some μ^K.*

Proof. Let $m \in (\hat{\mu}(r), \mathbf{E}H_0^r]$ for $r < \hat{r}(1)$, the boundary case follow from setting $\mu^K = 1$ when $m = 1$ and setting $\mu^K = 0,\ \mu_0 = 1$ when $m = \hat{\mu}(r)$.

Consider any q_1 for which there is an interval of types in $[\mu', \mu''] \subset [0, \mu_0]$ such that $\frac{K(\mu)}{u_1(r)} < q_1(\mu) < \frac{K(\mu)}{u_1(r) - \alpha_r(\mu) u_0(r)}$. Pick $\delta > 0$ such that $\mu'' - \mu' \geq 2\delta$. We can improve the revenue by increasing q_1 on $[\mu'' - \delta, \mu'')$ by some $\varepsilon_1 > 0$ and by reducing q_1 on $[\mu', \mu' + \delta)$ by $\varepsilon_0 > 0$ where

$$\varepsilon_1 = \varepsilon_0 \frac{\int_{\mu'}^{\mu'+\delta} \left((1-\mu) u_1(r) - \mu \frac{1-m}{m} u_0(r) \right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \left((1-\mu) u_1(r) - \mu \frac{1-m}{m} u_0(r) \right) dF(\mu)}$$

ensuring that the target mean, i.e. Equation 38 is maintained. This is possible as $(1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)$ is positive on $[0, \mu_0]$. We can choose ε_0 small enough such that the adjustment doesn't violate the constraint in Equation 19 as it weakly decreases the size of any point mass at $\hat{\mu}(r)$ without changing the target mean.

The revenue from the new allocation is greater than the old allocation if the following holds:

$$\frac{\int_{\mu'}^{\mu'+\delta} \left((1-\mu) u_1(r) - \mu \frac{1-m}{m} u_0(r) \right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \left((1-\mu) u_1(r) - \mu \frac{1-m}{m} u_0(r) \right) dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta} \mu \left(u_1(r) + u_0(r) \frac{1-m}{m} - \frac{R_0}{m} \right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu \left(u_1(r) + u_0(r) \frac{1-m}{m} - \frac{R_0}{m} \right) dF(\mu)}$$

$$\iff \frac{\int_{\mu'}^{\mu'+\delta} \left((1-\mu) u_1(r) - \mu \frac{1-m}{m} u_0(r) \right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \left((1-\mu) u_1(r) - \mu \frac{1-m}{m} u_0(r) \right) dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta} \mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu dF(\mu)}$$

$$\iff \frac{\int_{\mu'}^{\mu'+\delta} u_1(r) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} u_1(r) dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta} \mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu dF(\mu)}$$

$$\iff \frac{\int_{\mu'}^{\mu'+\delta} dF(\mu)}{\int_{\mu''-\delta}^{\mu''} dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta} \mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu dF(\mu)}$$

The first implication follows as $m \geq \hat{\mu}(r)$ implies that $u_1(r) + u_0(r)\frac{1-m}{m} - \frac{R_0}{m} \geq 0$. The second implication holds as $\mu \leq \mu_0 < \frac{u_1(r)}{u_1(r) + \frac{1-m}{m} u_0(\mu)}$ which implies that $(1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r) > 0$. The last inequality follows from our choice of δ.

Now consider q_1 such that there are intervals $I_1 < I_2 \subset [0, \mu_0]$ where $q_1(\mu) = \frac{K(\mu)}{u_1(r) - \alpha_r(\mu) u_0(r)}$ for $\mu \in I_1$ and $q_1(\mu) = \frac{K(\mu)}{u_1(r)}$ for $\mu \in I_2$. We construct an improvement similar to the above by slightly increasing q_1 on I_2 while reducing q_1 on I_1 to keep the mean constraint binding. Finally, note that this improvement introduces more slack to Equation 19. The revenue of this improvement is greater than the old allocation if the following holds:

$$\frac{\int_{I_1} dF(\mu)}{\int_{I_2} dF(\mu)} \geq \frac{\int_{I_1} \mu dF(\mu)}{\int_{I_2} \mu dF(\mu)}$$

This is implied by $\frac{\inf(I_2)}{\sup(I_1)} \geq 1$.

We can repeat the above arguments to establish that q_1 can not be such that $\frac{K(\mu)}{u_1(r)} < q_1(\mu) < \min\left\{1, \frac{K(\mu)+u_0(r)}{u_1(r)}\right\}$ for $\mu_0 \leq \mu \leq \frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}$. Moreover, there are no intervals $I_1 < I_2 \subset \left[\mu_0, \frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}\right]$ such that $q_1(\mu) = \min\left\{1, \frac{K(\mu)+u_0(r)}{u_1(r)}\right\}$ for $\mu \in I_1$ and $q_1(\mu) = \frac{K(\mu)}{u_1(r)}$ for $\mu \in I_2$.

To complete the proof consider $\mu > \frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}$. Assume for contradiction that there is an interval of types in $[\mu', \mu''] \subset [0, \mu_0]$ such that $\frac{K(\mu)}{u_1(r)} < q_1(\mu) < \min\left\{1, \frac{K(\mu)+u_0(r)}{u_1(r)}\right\}$. Pick $\delta > 0$ such that $\mu'' - \mu' \geq 2\delta$. We can improve the revenue by decreasing q_1 on $[\mu'' - \delta, \mu'')$ by some $\varepsilon_1 > 0$ and by increasing q_1 on $[\mu', \mu' + \delta)$ by $\varepsilon_0 > 0$ where

$$\varepsilon_1 = \varepsilon_0 \frac{\int_{\mu'}^{\mu'+\delta} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)}$$

Again, this adjustment can be made while maintaining Equation 38 and Equation 19.

The revenue from the new allocation is greater than the old allocation if the following holds:

$$-\frac{\int_{\mu'}^{\mu'+\delta} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \left((1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r)\right) dF(\mu)} \geq -\frac{\int_{\mu'}^{\mu'+\delta} \mu \left(u_1(r) + u_0(r)\frac{1-m}{m} - \frac{R_0}{m}\right) dF(\mu)}{\int_{\mu''-\delta}^{\mu''} \mu \left(u_1(r) + u_0(r)\frac{1-m}{m} - \frac{R_0}{m}\right) dF(\mu)}$$

$$\iff -\frac{\int_{\mu'}^{\mu'+\delta}\left((1-\mu)u_1(r)-\mu\frac{1-m}{m}u_0(r)\right)dF(\mu)}{\int_{\mu''-\delta}^{\mu''}\left((1-\mu)u_1(r)-\mu\frac{1-m}{m}u_0(r)\right)dF(\mu)} \geq -\frac{\int_{\mu'}^{\mu'+\delta}\mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''}\mu dF(\mu)}$$

$$\iff \frac{\int_{\mu'}^{\mu'+\delta}u_1(r)dF(\mu)}{\int_{\mu''-\delta}^{\mu''}u_1(r)dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta}\mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''}\mu dF(\mu)}$$

$$\iff \frac{\int_{\mu'}^{\mu'+\delta}dF(\mu)}{\int_{\mu''-\delta}^{\mu''}dF(\mu)} \geq \frac{\int_{\mu'}^{\mu'+\delta}\mu dF(\mu)}{\int_{\mu''-\delta}^{\mu''}\mu dF(\mu)}$$

The second implication holds as $\mu \geq \frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}$ which implies that $(1-\mu)u_1(r) - \mu\frac{1-m}{m}u_0(r) < 0$. The last inequality follows from our choice of δ.

Similarly, we can show that there are no intervals $I_1 < I_2 \subset \left[\frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}, 1\right]$ such that $q_1(\mu) = \min\left\{1, \frac{K(\mu)+u_0(r)}{u_1(r)}\right\}$ for $\mu \in I_2$ and $q_1(\mu) = \frac{K(\mu)}{u_1(r)}$ for $\mu \in I_1$. □

C.4 Optimal Choice of K

Lemma 9. *If $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ non-increasing, then there is a solution to the relaxed problem, q^K, for which $K(\mu) = 0$ on $\mu \leq \mu^K$.*

Proof. Consider $r < \hat{r}(1)$ and target mean $m \in [\hat{\mu}(r), \mathbf{E}H_0^r]$. In particular $K(\mu) \neq u_1(r)$ for all μ.

Let $\widetilde{\mu}$ be the largest type for which either $q_1^K(\mu) \neq 1$ or $q_1^K(\mu) \neq 0$. As $\mathbf{E}H_0^r < 1$ for $r < \hat{r}(1)$ it must be that $\widetilde{\mu} > \mu^K$.

If q_1^K is such that $K(\mu) \neq 0$ on $\mu \in [0, \mu^K)$ then $m \in (\hat{\mu}(r), \mathbf{E}H_0^r]$. Moreover, by monotonicity of K it must be that $\mu^* < \mu^K$. Where μ^* is the largest type for which $K(\mu) = 0$.

The allocation can be perturbed on an interval $[\widetilde{\mu} - \widetilde{\delta}, \widetilde{\mu})$ such that the new allocation is $(1, 0)$ on $[\widetilde{\mu} - \widetilde{\delta}, \widetilde{\mu})$. Where $\widetilde{\delta} > 0$ is arbitrarily small. For small enough $\widetilde{\delta}$ there exists $\delta, \delta^* > 0$ such that $\mu^* + \delta^* + \delta < \mu^K$ and the following holds

$$\int_{\mu^*}^{\mu^*+\delta^*} \mu q_1^K(\mu)\, dF(\mu) = \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \mu(1 - q_1^K(\mu))\, dF(\mu)$$

and

$$\int_{\mu^*+\delta^*}^{\mu^*+\delta^*+\delta} (1-\mu) \min\left\{1, \alpha_r(\mu^*+\delta^*)\frac{K(\mu^*+\delta^*)}{2u_1(r)}\right\} dF(\mu) = \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} (1-\mu) q_0^K(\mu)\, dF(\mu)$$

Thus, the allocation can be further perturbed on $[\mu^*, \mu^*+\delta+\delta^*)$ by allocating types in $[\mu^*, \mu^*+\delta^*)$ to $(0,0)$ and types in $[\mu^*+\delta^*, \mu^*+\delta+\delta^*)$ to $\left(\frac{K(\mu)}{u_1(r)}, \min\left\{1, \alpha_r(\mu^*+\delta^*)\frac{K(\mu^*+\delta^*)}{2u_1(r)}\right\}\right)$.

By construction, this perturbation preserves Equation 38, Equation 19 and Equation 41. Moreover, the revenue of the resulting perturbed allocation is greater than the original allocation q_1^K. The improvement in revenue follows from noting that non-decreasing $\frac{1-F(\mu)}{\mu f(\mu)}$ and $\frac{1-F(\mu)}{(1-\mu) f(\mu)}$ implies the following inequalities respectively

$$\begin{aligned}
&\int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \mu \left[(u_1(r)-R_0) - \frac{1}{\mu}\frac{1-F(\mu)}{f(\mu)} u_1(\mu)\right] q_1^K(\mu)\, dF(\mu) \\
&\qquad \geq \int_{\mu^*}^{\mu^*+\delta^*} \mu \left[(u_1(r)-R_0) - \frac{1}{\mu}\frac{1-F(\mu)}{f(\mu)} u_1(\mu)\right] (1-q_1^K(\mu))\, dF(\mu)
\end{aligned}$$

$$\begin{aligned}
\implies &\int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \left[\mu(u_1(r)-R_0) - \frac{1-F(\mu)}{f(\mu)} u_1(\mu)\right] q_1^K(\mu)\, dF(\mu) \\
&\qquad \geq \int_{\mu^*}^{\mu^*+\delta^*} \left[\mu(u_1(r)-R_0) - \frac{1-F(\mu)}{f(\mu)} u_1(\mu)\right] (1-q_1^K(\mu))\, dF(\mu)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} (1-\mu) \left[(u_1(r)-R_0) + \frac{1-F(\mu)}{(1-\mu)f(\mu)} u_1(\mu)\right] q_0^K(\mu)\, dF(\mu) \\
&\leq \int_{\mu^*+\delta^*}^{\mu^*+\delta^*+\delta} (1-\mu) \left[(u_1(r)-R_0) + \frac{1-F(\mu)}{(1-\mu)f(\mu)} u_1(\mu)\right] \min\left\{1, \alpha_r(\mu^*+\delta^*)\frac{K(\mu^*+\delta^*)}{2u_1(r)}\right\} dF(\mu)
\end{aligned}$$

$$\begin{aligned}
\implies &\int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \left[(1-\mu)(u_1(r)-R_0) + \frac{1-F(\mu)}{f(\mu)} u_1(\mu)\right] q_0^K(\mu)\, dF(\mu) \\
&\leq \int_{\mu^*+\delta^*}^{\mu^*+\delta^*+\delta} (1-\mu) \left[(u_1(r)-R_0) + \frac{1-F(\mu)}{f(\mu)} u_1(\mu)\right] \min\left\{1, \alpha_r(\mu^*+\delta^*)\frac{K(\mu^*+\delta^*)}{2u_1(r)}\right\} dF(\mu)
\end{aligned}$$

□

Lemma 10. *If* $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ *non-increasing, then there is a solution to the relaxed problem,* q^K*, for which* $K(\mu) = 0$ *on* $\mu \le \mu_0$.

Proof. By Lemma 9 we can restrict attention to allocations with $K(\mu) = 0$ on $\mu \le \mu^K$. Consider q_1^K such that $K(\mu) \ne 0$ on $[\mu^K, \mu_0]$ and $\mu^K < \mu_0$. As $\mu^K < \mu_0$ we get that the q^K corresponds to the relaxed problem with $x > 0$ and hence a target mean $m < \mathbf{E}H_0^r$.

Let μ^* be the largest type for which $K(\mu) = 0$. By monotonicity of K we get that $\mu^* < \mu_0$. For small enough $\delta^* > 0$ we can perturb the allocation on $(\mu^*, \mu^* + \delta^*]$ into the allocation $(0, 0)$. As types in $(\mu^*, \mu^* + \delta^*]$ are indifferent between participating or not, the perturbation does not affect the gross surplus and reduces the information rent by flattening the slope K on $(\mu^*, \mu^* + \delta^*]$. But the perturbation might violate Equation 18 and Equation 19 as the increase in mean from the perturbation might be above $\mathbf{E}H_{x'}^r$ where x' is the size of the point mass at $\hat{\mu}(r)$ of the perturbed allocation.

To restore these constraints, we further perturb the allocation by decreasing $q_0^K(\mu)$ by $\varepsilon > 0$ on $[\mu^* + \delta^*, \mu_0)$. The size of the mass point at $\hat{\mu}(r)$ jumps from x to 0 discontinuously as $\varepsilon > 0$. The mean from the two perturbations increases continuously in δ^* and ε. By combining the two perturbations Equation 18 and Equation 19 hold for $m' \in [m, \mathbf{E}H_0^r]$ and $x = 0$.

But the combination of these might violate the monotonicity of the slope. To address this, consider highest type $\widetilde{\mu}(\varepsilon) \ge \mu_0$ for which $K(\mu) \ge K(\mu_0) + u_0(r)\varepsilon$. By monotonicity of K, $\widetilde{\mu}(\varepsilon)$ is increasing in ε and equal to μ_0 at $\varepsilon = 0$. On $[\mu_0, \widetilde{\mu}(\varepsilon)]$ the allocation can be further perturbed by increasing $q_1^K(\mu)$ and/or decreasing $q_0^K(\mu)$ where these changes increase the slope for type μ from $K(\mu)$ to $K(\mu_0) + \varepsilon$. Note that this perturbation also increases the mean continuously in ε.

The combination of the three perturbations is feasible for the relaxed problem. The first perturbation does not affect the gross surplus. The second and the third perturbations increase the gross surplus. Moreover, ε and δ^* can be chosen such that the increase in the information rent from the second and third perturbation is offset by the decrease in information rent from the first perturbation. In particular, the following inequality holds

$$\int_{\mu^*}^{\mu^*+\delta^*} \frac{1-F(\mu)}{f(\mu)} K(\mu) dF(\mu) \\ \geq \int_{\mu^*+\delta^*}^{\mu_0} u_0(r)\varepsilon \frac{1-F(\mu)}{f(\mu)} dF(\mu) + \int_{\mu_0}^{\widetilde{\mu}(\varepsilon)} (K(\mu_0)+\varepsilon - K(\mu)) \frac{1-F(\mu)}{f(\mu)} dF(\mu)$$

The existence of this revenue-increasing and feasible perturbed allocation shows that q^K is not the solution to the relaxed problem, which establishes the lemma. □

For given K let μ_1 be the smallest type such that $K(\mu) \geq u_1(r) - u_0(r)$. Note that by definition $\mu_0 \leq \mu_1$. Combining Lemma 9 and Lemma 10 gives that if $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing, then there is a solution of the relaxed problem with one of the following form

$$q_1^K(\mu) := \begin{cases} 0 & \mu < \mu_0 \\ \frac{K(\mu)+u_0(r)}{u_1(r)} & \mu_1 > \mu \geq \mu_0 \\ 1 & \overline{\mu} > \mu \geq \mu_1 \\ \frac{K(\mu)}{u_1(r)} & \mu \geq \overline{\mu} \end{cases}$$

Lemma 11. *If $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ non-increasing, then there is a solution to the relaxed problem, q^K, for which K is constant on $[\overline{\mu}, 1]$ and is equal to $\lim_{\mu \uparrow \overline{\mu}} K(\mu)$ or 1.*

Proof. If K changes in value on $[\overline{\mu}, 1)$, by monotonicity of K there exists $\widetilde{\mu}$ in $[\overline{\mu}, 1)$ and $\delta^* > 0$ such that $K(\overline{\mu} + \delta^*) < K(\widetilde{\mu})$. In particular $K(\overline{\mu}) \leq K(\overline{\mu} + \delta^*) < u_1(r)$.

This implies that for small enough $\varepsilon > 0$ the allocation resulting from perturbing q^K on $[\overline{\mu},\ \overline{\mu} + \delta^*)$ by $+\left(\varepsilon,\ \varepsilon\frac{u_1(r)}{u_0(r)}\right)$ for small enough $\varepsilon > 0$ preserves the slope K, Equation 14 and Equation 19.

This perturbation, however, increases the mean, and the new allocation can thus violate Equation 18. To restore the target mean, the allocation can be further perturbed by increasing q_0^K slightly for all types μ in $[\widetilde{\mu}, 1)$. Let $\varepsilon' > 0$ be the amount of this increase, and satisfies the following condition

$$\varepsilon' \int_{\overline{\mu}}^{\widetilde{\mu}} dF(\mu) = \varepsilon \int_{\overline{\mu}}^{\overline{\mu}+\delta^*} \mu \frac{1-m}{m} - (1-\mu)\frac{u_1(r)}{u_0(r)}\, dF(\mu)$$

The right hand side is positive as $\overline{\mu} \geq \frac{u_1(r)}{u_1(r)+\frac{1-m}{m}u_0(\mu)}$. The value of $\varepsilon' > 0$ is arbitrarily small by choice of δ^* and ε, and hence Equation 14 and Equation 19 hold. By construction Equation 18 also holds. The first statement of the claim follows from noting that the combination of the two perturbations increases the gross surplus and decreases the rents.

To establish that K equals $\lim_{\mu \uparrow \overline{\mu}} K(\mu)$ or 1, we can then use our usual argument of reducing the allocation q_1^K on $[\mu_0, \mu_0 + \delta)$ and increasing q_1^K on an interval $[1-\widetilde{\delta}, 1]$ for some $\delta, \widetilde{\delta} > 0$ while preserving the mean constraint Equation 18. Like before, we can show that as $\frac{1-F(\mu)}{\mu f(\mu)}$ non-increasing, this perturbation is revenue improving as it reduces the information rent. □

Lemma 12. *If $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ non-increasing, then there is a solution to the relaxed problem, q^K, for which K is constant on $[\mu_0, \mu_1)$ and is equal to $\alpha_r^{-1}(\mu_0)u_1(r) - u_0(r)$.*

Proof. From Lemma 9 and Lemma 10 it suffices to consider q^K such that $K(\mu) = 0$ for all types in $[0, \mu_0)$ Let $\widetilde{\mu}$ be the smallest type for which $q_1^K(\mu) = 1$. If $K(\mu)$ is not constant and equal to $\alpha_r^{-1}(\mu_0)u_1(r) - u_0(r)$ in $[\mu_0, \mu_1)$ by Equation 14 we have $\frac{K(\mu_0)+u_0(r)}{u_1(r)} > \alpha_r^{-1}(\mu_0)$. Thus there exist arbitrarily small δ^*, $\widetilde{\delta} > 0$ such that $\mu_0 + \delta^* < \widetilde{\mu} - \widetilde{\delta}$ and

$$\int_{\mu_0}^{\mu_0+\delta^*} \mu \left(\frac{K(\mu) + u_0(r)}{u_1(r)} - \alpha_r^{-1}(\mu_0) \right) dF(\mu) = \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \mu(1 - q_1^K(\mu))\, dF(\mu)$$

The allocation can be perturbed by decreasing $q_1^K(\mu)$ to $\alpha_r^{-1}(\mu_0)$ on $[\mu_0, \mu_0 + \delta^*]$ and by increasing $q_1^K(\mu)$ to 1 on $[\widetilde{\mu} - \widetilde{\delta},\, \widetilde{\mu})$. This preserves Equation 18, Equation 14 and Equation 19. The claim follows from noting that the perturbation weakly increases revenue as

$$\int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \mu \left[(u_1(r) - R_0) - \frac{1}{\mu}\frac{1-F(\mu)}{f(\mu)} u_1(\mu) \right] \left(\frac{K(\mu)+u_0(r)}{u_1(r)} - \alpha_r^{-1}(\mu_0) \right) (\mu)\, dF(\mu)$$
$$\geq \int_{\mu^*}^{\mu^*+\delta^*} \mu \left[(u_1(r) - R_0) - \frac{1}{\mu}\frac{1-F(\mu)}{f(\mu)} u_1(\mu) \right] (1 - q_1^K(\mu))\, dF(\mu)$$

$$\implies \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} \left[\mu(u_1(r) - R_0) - \frac{1-F(\mu)}{f(\mu)} u_1(\mu)\right] \left(\frac{K(\mu) + u_0(r)}{u_1(r)} - \alpha_r^{-1}(\mu_0)\right) dF(\mu)$$
$$\geq \int_{\mu^*}^{\mu^*+\delta^*} \left[\mu(u_1(r) - R_0) - \frac{1-F(\mu)}{f(\mu)} u_1(\mu)\right] (1 - q_1^K(\mu))\, dF(\mu)$$

□

Lemma 13. *If $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ non-increasing, then there is a solution to the relaxed problem, q^K, for which K equals either $K(\mu_1)$ or $K(\overline{\mu})$ on $[\mu_1, \overline{\mu})$.*

Proof. If $K(\mu) \in (K(\mu_1), K(\overline{\mu}))$ on $[\mu_1, \overline{\mu}]$ we can construct a profitable perturbation by decreasing $q_0^K(\mu)$ by $\frac{K(\overline{\mu})-K(\mu)}{u_0(r)}$ on $[\overline{\mu} - \widetilde{\delta}, \overline{\mu})$ and by increasing $q_0^K(\mu)$ by $\frac{K(\mu)-K(\mu_1)}{u_0(r)}$ on $[\mu_1, \mu_1 + \delta^*)$. Where $\widetilde{\delta}, \delta^* > 0$ are such that $\mu_1 + \delta^* < \overline{\mu} - \widetilde{\delta}$ and satisfy the following

$$\int_{\mu_1}^{\mu_1+\delta^*} (1-\mu) \frac{K(\overline{\mu}) - K(\mu)}{u_0(r)}\, dF(\mu) = \int_{\widetilde{\mu}-\widetilde{\delta}}^{\widetilde{\mu}} (1-\mu) \frac{K(\mu) - K(\mu_1)}{u_0(r)}\, dF(\mu)$$

This perturbation preserves incentive compatibility and obedience and yields a greater revenue as $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing. □

C.5 Proof of Proposition 3

Using Lemmas 7 to 12, we observe that to prove the proposition, it suffices to pin down the value of $K(\mu)$ on $[\mu_1, \overline{\mu})$ and whether $\mu_0 \geq \hat{\mu}(r)$ or not.

If K changes values on $[\mu_1, \overline{\mu})$ then by monotonicity of K and Lemma 13 there exists $\mu^* \in (\mu_1, \overline{\mu})$ such that $K(\mu) = K(\mu_1) < K(\overline{\mu})$ for $\mu \in [\mu_1, \mu^*)$.

If $\overline{\mu} < 1$ then it must be that $K(\overline{\mu}) = u_1(r)$ as otherwise we can construct a perturbation similar to Lemma 11 by perturbing q^K on $\left[\overline{\mu},\, \overline{\mu} + \widetilde{\delta}\right)$ by $+\left(\varepsilon,\, \varepsilon \frac{u_1(r)}{u_0(r)}\right)$. For small enough $\varepsilon > 0$ this perturbation preserves the slope K, Equation 14 and Equation 19. Like Lemma 11, this perturbation increases the mean, and the new allocation can thus violate Equation 18. To restore the target mean, the allocation can be further perturbed by increasing q_0^K slightly for all types μ in $[\mu^*, \overline{\mu})$. Let $\varepsilon' > 0$ be the amount of this increase, and satisfies the following conditions

$$K(\overline{\mu}) - \varepsilon' u_0(r) \geq K(\mu_1)$$

and

$$\varepsilon' \int_{\mu^*}^{\overline{\mu}} (1-\mu) dF(\mu) = \varepsilon \int_{\overline{\mu}}^{\overline{\mu}+\widetilde{\delta}} \mu \frac{1-m}{m} - (1-\mu) \frac{u_1(r)}{u_0(r)} \, dF(\mu)$$

If $\overline{\mu} = 1$, then by our usual manipulation, a mean-preserving, revenue-improving, and feasible perturbation can be constructed by increasing q_0^K slightly for all types μ in $[\mu^*, \mu^* + \hat{\delta})$ and slightly decreasing q_0^K on $[1 - \delta'', 1]$. Where $\hat{\delta}, \delta > 0$ can be arbitarily small.
Thus $K(\overline{\mu}) = u_1(r)$ whenever K changes value in the interval $[\mu_1, \overline{\mu})$

If $\hat{\mu}(r) \leq \mu_0 < \mu_1 < \overline{\mu} < 1$ then $K(\overline{\mu}) = u_1(r)$. Where $K(\overline{\mu}) = u_1(r)$ follows by noting that if $K(\overline{\mu}) < u_1(r)$ then we can construct a mean-preserving perturbation of the allocation by increasing $q_1^K(\mu)$ to 1 on $[\widetilde{\mu}, 1] \subset [\overline{\mu}, 1]$ and by decreasing q_1^K to $\alpha_r^{-1}(\mu_0) u_1(r) - u_0(r)$ on $[\mu_1, \mu_1 + \delta^*]$ for $\delta^* > 0$. This increases revenue by our usual manipulation and by noting that $\frac{1-F(\mu)}{\mu f(\mu)}$ is non-increasing.
Moreover, it must be that $K(\mu_1) \in \{u_1(r) - u_0(r), u_1(r)\}$. To see this note that if for contradiction $K(\mu_1) \in (u_1(r) - u_0(r), u_1(r))$. Then we can increase $q_0^K(\mu)$ to 1 on $[\mu_1, \mu_1 + \delta)$ and decreasing $q_0^K(\mu)$ to 0 on $[\overline{\mu} - \delta', \overline{\mu})$. Where $\delta, \delta' > 0$ and $\mu_1 + \delta < \overline{\mu} - \delta'$. This adjustment can be made to preserve Equation 18 and is revenue improving when $\frac{1-F(\mu)}{(1-\mu) f(\mu)}$ is non-increasing.

Finally, consider $\mu_1 = \mu_0$. From the previous arguments we can deduce that if $K(\overline{\mu}) = u_1(r)$ then $K(\mu)$ is constant on $[\mu_1, \overline{\mu})$ and is equal to $u_1(r) - u_0(r)$ or $u_1(r) - \alpha_r(\mu_0) u_0(r)$. Moreover, if $K(\overline{\mu}) < u_1(r)$ then $K(\mu)$ is constant and equal to $u_1(r) - \alpha_r(\mu_0) u_0(r)$ on $[\mu_1, 1]$.

C.6 Optimal Mean

Lemma 14. *Fix some $r \in [\hat{r}(0), \hat{r}(1)]$. If $\frac{1-F(\mu)}{(1-\mu) f(\mu)}$ is non-increasing, then there is a solution to the relaxed problem which has a form given in Proposition 3 and is such that* *either Equation 18 holds with $m = \mathbf{E}H_0^r$ or the allocation has the following form*

$$q(\mu) = \begin{cases} (0,0) & \mu < \mu_1 \\ \left(\frac{u_0(r)}{u_1(r)}, 1\right) & \mu_1 > \mu \geq \mu_0 \\ (1,1) & \overline{\mu} > \mu \geq \mu_1 \\ (1,0) & \mu \geq \overline{\mu} \end{cases}$$

Where $\alpha_r^{-1}(\mu_1) < 1$ if $\mu_0 = \mu_1$.

Proof. Let q be an allocation with one of t forms from Proposition 3.

First consider the allocation such that $\overline{\mu} < 1$ and $q_1 < 1$ for $\mu \geq \overline{\mu}$. If in this case $m < \mathbf{E}H_0^r$, we can construct a profitabler deviation similar to Lemma 11 by increasing the allocation q by $+(\varepsilon, \varepsilon\frac{u_1(r)}{u_0(r)}$ for small $\varepsilon > 0$. This perturbation does not affect the slope of the indirect utility and hence leaves the information rent unchanged. The perturbation increases the mean of the allocation as $\overline{\mu} \geq \frac{u_1(r)}{u_1(r) + \frac{1-m}{m}u_0(r)} \geq \hat{\mu}(r)$ and hence increases the gross surplus. For small enough ε, Equation 19 and Equation 14 hold for the perturbed allocation. Moreover, the mean m' of the perturbed allocation is in $(m, \mathbf{E}H_0^r)$. Thus, the perturbed allocation is feasible for the relaxed problem and yields a greater revenue, leading to a contradiction.

Now consider allocation such that $\mu_0 < \mu_1, \overline{\mu}$ and $\alpha_r^{-1}(\mu_0) > \frac{u_0(r)}{u_1(r)}$. For such an allocation the type μ_0 contributes 0 to the gross surplus, moreover for small enough $\delta > 0$ the contribution of types in $[\mu_0, \mu_0 + \delta)$ contribute an arbitrarily small amount to the gross surplus. When $\alpha_r^{-1}(\mu_0) > \frac{u_0(r)}{u_1(r)}$ the indirect utility of types in $[\mu_0, \mu_1)$ is strictly greater than zero. In particular $u_1(r)q_1(\mu_0) - u_0(r)q_0(\mu_0) > 0$. Thus we can choose $\delta^* > \delta > 0$ small enough such that for all $\nu \in [\mu_0, \mu_0 + \delta)$ the following holds

$$\nu(u_1(r) - R_0)q_1(\mu_0) + (1-\nu)(u_1(r) - R_0)q_0(\mu_0) \\ < \\ \max\left\{\frac{1 - F(\mu_0 + \delta^*)}{(1 - \mu_0 + \delta^*)f(\mu_0 + \delta^*)}, \frac{1 - F(\mu_0 + \delta^*)}{(\mu_0 + \delta^*)f(\mu_0 + \delta^*)}\right\}[u_1(r)q_1(\mu_0) - u_0(r)q_0(\mu_0)]$$

Thus, we can construct a revenue-improving and feasible perturbation by changing the allocation to $(0, 0)$ on $[\mu_0, \mu_0 + \delta)$ and unchanged otherwise. The revenue improves as the above inequality implies that the reduction in information rent from the perturbation is greater than the reduction in the gross surplus. The perturbed allocation is feasible as it leaves Equation 19 and Equation 14 unchanged and for a small enough $\delta > 0$ the new mean is $m' \in (m, \mathbf{E}H_0^r)$.

$\square$