Employer Competition and Certification

Hershdeep Chopra [*]

Northwestern University

Abstract

This paper develops a theory of employer competition over *hiring standards* in labor markets where employers rely on third-party certification to screen applicants. A revenue-maximizing certifier sells tests to an applicant, who possesses imperfect private information about his ability and seeks to persuade employers to offer him employment. The certifier faces a joint screening and information design problem in designing a test allocation. The distortions from screening reduce the overall informativeness of the test allocation, steering the applicant supply towards less selective employers. This incentivizes the more selective employers to lower their standards, intensifying employer competition.

KEYWORDS: Monopoly Certification, Information Acquisition, Mechanism Design, Communication Game, Adverse Selection, Labor Markets

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1 Introduction

Labor markets are subject to information asymmetries as applicants generally know more about their skills than potential employers. This asymmetry can lead to market failure if the applicant cannot credibly signal his private information and if the expected ability of the applicant is low, as in Akerlof (1970). An institutional response to market failure is third-party certification. The certifier allows the informed applicant to signal information to uninformed employers (decision makers). The certifier sells tests to the applicant, whose outcome may depend on the applicant's

[*]Department of Economics, Northwestern University. hershdeepchopra2026@u.northwestern.edu.

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underlying ability. The test outcome informs the employers about the applicant's ability, alleviating some of the asymmetry.

Physicians, financial advisers, teachers, lawyers, and other professionals are required to take standardized tests. Some other examples of such tests include online skill certification through labor market platforms like Freelancer.com and talent assessment firms like ExpertRating. Employers often screen applicants through an established minimum *hiring standard*.[1] These range from pass/fail requirements, such as the Uniform Bar Exam for prospective lawyers, to raw score requirements, as with the USMLE (United States Medical Licensing Examination) for hospital residency programs. In entry-level professions, where wages are standardized, employers compete in their selectivity to attract applicants. More sought-after employers can aggressively screen applicants by having higher hiring standards.

The institutions responsible for administering these tests face an incentive problem in allocating tests to privately informed applicants. Rent-extraction by the certifier can lead to distortions in test allocation, which feeds back into the employers' decisions. To parse the effect of the certifier's rent-extracting motive on the test allocation and on the subsequent employer-applicant interaction, I abstract away from institutional details and consider a revenue-maximizing certifier that can flexibly design and price information through menu pricing of tests.[2]

In that respect, this paper connects two elements that are typically studied in isolation: the market structure for allocating tests to the applicant and the downstream employer-applicant interaction. This provides new theoretical insights about the connection between allocative efficiency in the certifier-applicant market and the nature of competition between employers. In particular, the distortions from second-degree price discrimination by the certifier reduce the overall informativeness of the test allocation.[3] This tends to increase the supply of applicants to less selective employers, which incentivizes the more selective employers to lower their *hiring standards*.

Rent-extraction by the certifier coarsens the overall information conveyed by the test allocation to the employers, leading to a lower supply of high-credential applicants and consequently hindering employers from aggressively screening applicants.

Certification and Standards: The certifier can be viewed as a two-sided supplier. The certifier sells tests to a privately informed applicant. The certifier "indirectly supplies" certified applicants

[1]Hiring standard represents the minimal expected ability of an applicant that an employer is willing to hire.

[2]Instances of such rent-extracting certification are tiered/nested testing structures in licensing finance professionals and for-profit talent assessment firms like ExpertRating or TestGorilla.

[3]Lizzeri (1999), Kartik et al. (2021) show monopolistic certification, without screening, has an *informativeness-reducing* effect. I expand on this insight by showing that second-degree price discrimination by the certifier further reduces informativeness. The certifier increases allocation to less selective employers, allowing it to pool more low-ability applicants and thus reducing information rents.

to employers.[4] The certifier can utilize menus of tests to (second-degree) price discriminate the privately informed applicant. An employer makes the hiring decision based on the applicant's testing outcome and the menu of testing options available to the applicant. Tests contain hard information, as the outcomes depend on the applicant's underlying ability. Tests also contain soft information owing to the applicant's self-selection into different testing options made available by the certifier.

Employers can set high standards to drive out lower ability applicants when testing is sufficiently informative and costly. Although a convenient screening tool, an excessively high hiring standard can be counterproductive when employers compete for a limited pool of applicants. A competing employer might be able to poach potential high ability applicants by undercutting an employer with stricter standards. These considerations highlight the strategic role of hiring standards in addition to the aforementioned screening role.

Another strategic aspect of hiring standards is their ability to influence the tests designed by the certifier. As the applicant's willingness to pay for a test depends on the employers' hiring decisions, the hiring standards also affect the demand for certification tests.[5] This way, employers' hiring standards and the test allocation are jointly determined in an equilibrium. Due to this feedback, inefficiencies in test allocation can lead to unexpected consequences for employer competition.

Modeling Preliminaries: I consider a monopolistic certifier (test designer), an applicant who is partially and privately informed about his ability, and two employers that demand certified applicants and are differentiated by their reputation.[6,7]

The model abstracts away from a wage-setting mechanism to focus on certification's impact on employer competition. Such considerations also have practical merit; entry-level workers or workers in regulated professions face similar wages across employers, but they might prefer some employers over others due to wage-independent aspects like job mobility or workplace environment.[8]

[4] An alternate setting could allow the certifier to charge a price to the employers for revealing an applicant's test outcome. In this case, the certifier can be viewed as a direct supplier to the employer. I comment on this in section 7.2.

[5] Information is valuable only through the decisions it induces.

[6] Many examples of professional certification, including the ones alluded to above, involve third-party certification that is not necessarily monopolistic. Yet the monopolistic certification is an economically relevant benchmark, and insights from the analysis are useful for other settings as well.

[7] Restricting employers to a duopoly is for expositional ease; the results and techniques readily generalize to more employers.

[8] In section 7.1, I present an extension with wages. The main insight there is a negative result. If the certifier can flexibly design and price information (tests), then vertical differentiation of employers alone does not lead to meaningful wage competition.

I model imperfect employer competition through a vertically differentiated duoposony. I consider two employers that differ in their utility to the applicant upon being hired. I call them the top employer and the bottom employer. The utility from joining the top employer is greater than that from the bottom employer, and this is the same across all applicants. Employers rely on test outcomes to assess the ability of a potential applicant. Both employers have the same value for an applicant of a given ability; employers have a higher utility for higher ability applicants.

To focus on employer competition, I assume employers have greater bargaining power than the certifier. In particular, employers can commit to a hiring standard rather than choosing their standards in response to the mechanism designed by the certifier. Employers affect the demand for certified applicants by committing to hiring standards.

The applicant is partially privately informed about his ability. I refer to the applicant's belief about his ability is his type. As test outcomes depend on ability, the applicant's value for any given test varies with the belief about his ability. In particular, the applicant's type determines his willingness to pay for a test.

The employers and the certifier do not observe the applicant's type (and ability), and have a common prior F. I assume that in the absence of the certifier, there is a market failure, as the applicant is always left unemployed. In particular, I assume that the prior expected ability of the applicant is below the minimum expected ability of the applicant that the employers are willing to hire. This makes the certifier's ability to generate hard information (ability-contingent signaling) central in preventing market failure.

Due to uncertainty about the applicant's type, the certifier faces a screening problem when selling tests to the applicant. Tests are multi-dimensional instruments, each comprising a collection of outcome distributions indexed by the applicant's underlying ability.[9] The variation in a test's value across different types of applicants depends on the variation in the test's ability contingent outcome distributions. I restrict attention to binary ability to simplify the analysis of the screening problem faced by the certifier. The applicant is either high ability or low ability. The applicant's type is then captured by the probability he assigns to being high ability. Types are single-dimensional, sidestepping intricacies associated with the design of selling mechanisms in the presence of multi-dimensional type spaces.

Along with the screening problem above, the certifier faces an information design problem. Tests are valuable to an applicant only if they persuade the employer to hire the applicant. Hiring standards constrain the information a test must generate to persuade an employer. This restricts the certifier's ability to pool high and low ability applicants, thus constraining the surplus that the certifier can generate. The certifier needs to trade off surplus generation and information rents from screening friction, while conforming to hiring standards set by the employers.

[9] In economics and statistics literature such tests are often referred to as Blackwell (or statistical) experiments.

The interaction of the screening and information design problems results in inefficiencies that are characteristic of certification markets. The certifier reduces the informativeness of tests by pooling low ability applicants with high ablity applicants to extract more surplus from the applicant. As the certification mechanism and hiring standards are equilibrium objects, these inefficiencies spill over onto the employers' decisions.

Results: In the model, screening frictions faced by the certifier lead to two distortions relative to a benchmark where the certifier observes the applicant's type at the time of contracting. The benchmark captures the information design part of the certifier's problem (see section 4).

The first distortion is *exclusion*; there is an increase in the chance that the applicant is left unemployed. When facing a privately informed applicant, the certifier resorts to second-degree price discrimination. This involves offering a menu of different testing options at varying prices. The optimal selling mechanism for tests leads to exclusion whenever the expected ability of the applicants is low enough. In the presence of screening frictions, the designer does not sell to some types, increasing unemployment.[10] For exclusion to occur, it's crucial that the certifier needs to generate informative tests to induce employment. There is an excessive reduction in labor supply, relative to the benchmark, indicating that the inefficiency in test allocation amplifies certification's role as a barrier to entry for the applicant.

The second distortion is *reduction in informativeness*; the certifier distorts the test towards allocating the applicant to less selective employers with a greater probability. Less selective employers set lower hiring standards and are willing to hire applicants with lower expected ability. Pooling a larger quantity of low ability applicants helps reduce overall information rents, as it reduces the difference (heterogeneity) in test outcomes for high and low ability applicants. The certifier benefits from a greater mass of low ability applicants not only through changes in the applicant's gross utility but also through reductions in information rents conceded to the applicant. This *informativness reducing* effect of second-degree price discrimination is highlighted by Theorem 1 in section 5.2. Theorem 1 shows that whenever the top employer sets a higher standard than the bottom employer, the certifier allocates the applicant to the bottom employer with positive probability (and sometimes allocates the applicant only to the bottom employer). This contrasts with the benchmark, where the certifier allocates the applicant only to the top employer against a subset of these standards.

Although tests that allocate the applicant to the top employer fetch higher prices, they also raise information rents disproportionately when the top employer sets a higher standard than the bottom employer. Higher standards result in a smaller quantity of low ability applicants, leading

[10] This distortion is reminiscent of exclusion results from the literature on non-linear pricing, see Armstrong (1996) for example.

to greater heterogeneity in willingness to pay across applicant types and thus greater information rents. When the bottom employer sets a low enough standard, the certifier might prefer allocating the applicant to the bottom employer, even when it reduces the applicant's gross utility. This highlights an increase in the bottom employer's market power vis-à-vis the top employer, apparent in the relative selectivity of the employers. Compared to the benchmark, the gap between the hiring standards set by the top and the bottom employers becomes narrower, leading to *constriction of standards*.

I demonstrate this effect on employer competition by Theorem 2. In the benchmark allocation, if the bottom employer is *weak* (the utility that the applicant gets from joining the employer is low enough), then the equilibrium certification mechanism and top employer's hiring standards are independent of the bottom employer's choice. In contrast, when the certifier has to screen applicants, the equilibrium certification mechanism and top employer's standards might depend on the (*weak*) bottom employer's choice. Due to the *informativeness reducing* effect of second-degree price discrimination, the certifier might allocate applicants even to a *weak* bottom employer. This incentivizes the top employer to lower its standards. Theorem 2 shows that for some parameter values, the top employer never sets a standard of 1.[11] For the same parameter values, the unique equilibrium of the benchmark requires the top employer to set a standard of 1.

Relative to the benchmark, screening friction faced by the certifier results in test allocation that generates a lower gross surplus in the certifier-applicant (upstream) market. Both exclusion and reduction in informativeness hurt the top employer, but the bottom employer might benefit from the latter. To summarize, inefficiency in test provision leads to an overall loss of surplus in the market, but remarkably, **competition among the employers intensifies**.

The results of this paper identify the role of monopolistic certification in shaping employer competition by excessively reducing the supply of high-credential applicants through reducing the informativeness of test allocation.

2 Literature

Theoretical and empirical literature on certification is vast; an early contribution by Viscusi (1978) points out the role of certification in preventing market breakdown. In a seminal paper, Lizzeri (1999) explores the role of a monopolistic certifier who can sell information to privately informed parties. But the designer in Lizzeri (1999) uses a take-it-or-leave-it offer instead of a menu of tests and prices. Lizzeri (1999) assumes that markets do not unravel when the certifier is absent; the prior expected ability of the applicant is above the employers' reservation. Thus, restricting the

[11] If an employer sets a standard of 1, then the employer only accepts an applicant if it is certain that the applicant is of high ability.

certifier to a single take-it-or-leave-it offer is without loss as the certifier offers an uninformative test sold to all types and extracts all the surplus from the applicants. Kartik et al. (2021) generalizes the findings of Lizzeri (1999) to partially informed agents; they do not allow price discrimination by the certifier. A key insight in these papers is that a monopolistic certifier favors less informative tests. I extend the insight of these papers by showing that second-degree price discrimination, by the certifier, amplifies the economic force favoring less information.

In an influential paper Leland (1979) develops a theory of professional licensing in an environment where applicants invest in their ability. He shows that licensing, in the form of minimum ability standards, can prevent market failure. Leland (1979) does not consider certification or an information intermediary in his model; the choice and enforcement of standards is exogenous.

A well-known consequence of certification is its function as a barrier to entry (Stigler (1971)). Certification requirements create restrictions on the supply of applicants and potentially drive up competition between employers. I argue that inefficiency in test allocation, from certifier screening the applicant, not only further restricts the supply of applicants but also reduces the overall informativeness of the tests.

Dranove and Jin (2010) surveys, among other aspects of certification, the economic debate about certification's role in *quality assurance* versus its role as a *barrier to entry*. Naidu and Posner (2022) surveys the challenges present in regulating employer competition. Azar and Marinescu (2024) surveys recent developments in the theory of employer competition. They focus on three modeling approaches for employer competition – oligopoly, job differentiation, and search frictions. I study an oligopolistic labor market to address a fundamentally different question, focusing on the quality assurance role. How do distortions from screening affect employer competition when the certifier flexibly designs and prices information?

Following the works of Rayo and Segal (2010) and Kamenica and Gentzkow (2011), there has been an explosion of interest in studying information provision involving general information structures. This has led to new insights about markets and methodological advances Roesler and Szentes (2017), Bergemann et al. (2018), Kleiner et al. (2021), Dworczak and Kolotilin (2024). In a recent survey Bergemann and Ottaviani (2021) describes various market mechanisms for information provision. Using techniques from the information design literature, there have been recent developments in the study of certification, where the certifier produces hard information. Ali et al. (2020) has considered robust provision of hard information when the agent does not hold private information. Asseyer and Weksler (2024) also considers the provision of hard information to uninformed agent; they focus on a common value environment.

Combining mechanism design and information design problems has led to many interesting insights in various applications. For example, Calzolari and Pavan (2006), Dworczak (2020) have studied sequential agency problems in which upstream designers reveal strategic information to downstream principals. Bergemann and Pesendorfer (2007), Bergemann et al. (2022) have considered joint design of product allocation and information about the product. Frankel (2021) considers a delegation model in a labor market environment where employers get hard and soft information about applicants. The important difference is that the employer can observe the applicant's hard information, and the soft information is provided by a third agent (manager). Corrao (2023) has studied screening by a monopolistic certifier, but restricts attention to soft information.

There has been some recent interest and progress in studying certification intermediaries that generate new information and interact with partially informed buyers Weksler and Zik (2025), Celik and Strausz (2025), Mäkimattila et al. (2025), Chopra and Ely (2025). In Weksler and Zik (2025), the agents seeking certification are privately and partially informed, but their choice of tests is publicly observable. Celik and Strausz (2025) focuses on the role of soft and hard information (they call it screening and acquisition) in certification mechanisms for a buyer seller framework. Mäkimattila et al. (2025) focuses on monopolistic certification, similar to my single employer benchmark, and contrasts it with the setting in which the applicant's test choice is observable. In a related single employer model Ely (2025) studies optimal test allocation when the applicant can credibly reveal his private type to the employer (decision maker).

The closest paper to this one is Chopra and Ely (2025), which develops analytical tools for the mechanism and information design problems faced by the certifier in various contexts. Although both Mäkimattila et al. (2025) and Chopra and Ely (2025) describe the single employer analogue of the intermediaries problem (see section 6.1), neither considers the equilibrium interaction of the information receivers' commitment and certification mechanism.

3 Model

There are four players: an applicant, a top employer, a bottom employer, and a certifier (test designer). The applicant has ability $\theta \in \{h, l\}$, unknown to all. The applicant has partial private information about his ability represented by his **type**, $\mu \in [0, 1]$. Only the applicant knows his prior type. The employers and the certifier are initially uninformed about the applicant's ability θ and his type μ, and have a common prior $F \in \Delta([0, 1])$ with full support and a continuous pdf f.

Each employer receives a payoff of $v_h > 0$ from employing a high ability applicant and a payoff of $v_l < 0$ from employing a low ability applicant. The value v_θ can be understood as the productivity of an applicant with ability θ. The expression $\nu v_h + (1 - \nu)v_l$ is the expected productivity (value), to the employers, of an applicant with expected ability ν. The payoff from employing someone with expected ability ν is $\nu v_h + (1 - \nu)v_l$. Thus, the expected value of an applicant to the employers is increasing in the applicant's type. The employers are indifferent between hiring or rejecting an applicant with type $\mu = \frac{-v_l}{v_h - v_l}$. The cutoff $\frac{-v_l}{v_h - v_l}$ is the employers' *reservation* expected ability of the applicant.

I will assume that $1 > \frac{-v_l}{v_h - v_l} > \mathbf{E}_F[\ \mu\ ]$. This makes the role of certifier more interesting, as without the certifier, there is no employment. The assumption is to emphasize the certifier's role in information generation, in contrast to the gatekeeping role of certification. If $\mathbf{E}_F[\mu] < \frac{-v_l}{v_h - v_l}$ then the certifier enables credible signaling which prevents market failure. The certifier is a gatekeeper when $\mathbf{E}_F[\mu] \geq \frac{-v_l}{v_h - v_l}$. In this case, markets do not unravel without the certifier, as the applicant's prior expected value to the employers is positive.

Imperfect competition between employers is modeled through vertical differentiation. The applicant gets a utility of 1 from being hired by the top employer, a utility of $0 < u < 1$ for the bottom employer, and 0 if left unemployed. The differentiation captures the applicant's wage independent consideration, like the prestige (reputation) of the employer. The applicant can only be employed by one employer.

The certifier facilitates signaling through selling tests to the applicant. The certifier seeks to maximize its revenue from selling tests to the applicant. A test is defined by a set of test outcomes $\mathcal{M}_E$ and for each θ, a distribution $\rho_\theta \in \Delta(\mathcal{M}_E)$. When the applicant takes a test, the outcomes are drawn from ρ_θ when the applicant's ability is θ. The test outcome $m \in \mathcal{M}_E$ is revealed publicly. The certifier sets a (certification) *mechanism* Φ, which consists of a set of reports $\mathcal{M}$ and a function from elements of $\mathcal{M}$ to pairs of payments and tests. To allow for voluntary participation by the applicant, I require that the "empty message" is always available to the applicant at no cost.

Timing

1. The top and the bottom employers simultaneously and publicly **commit** to hiring standards $s = (s_t, s_b) \in [0,1]^2$.
2. The certifier publicly announces a certification mechanism Φ.
3. The applicant chooses a report in $\mathcal{M}$ and makes payment to the certifier as prescribed by Φ.
4. Employers observe the certification mechanism and the test outcome $m \in \mathcal{M}_E$.[12]
5. The applicant chooses an employer among those whose posterior belief based on 4. exceeds their hiring standards, and remains unemployed otherwise.

The solution concept is *Perfect Bayesian Equilibrium*, henceforth referred to as just equilibrium.

[12]I require that all employees can observe the test outcome. We can also allow the certifier to reveal test outcomes to subsets of employers. The certifier does not gain from this extra contractual power, as on-path payoffs in revenue-maximizing equilibria remain unchanged. See obedience constraints in section 5 for details.

Employers commit to the expected ability of an applicant that they are willing to accept (hiring standards). This commitment assumption helps focus on employer competition, avoiding considerations related to inter-market competition between the certifier and the employers.[13]

In a *direct* mechanism $\mathcal{M} = [0,1]$ and $\mathcal{M}_E = \Delta(A)$. The space of test outcomes (or recommendations) is $A = \{a_t, a_b, r\} := \{\text{hire by top employer, hire by bottom employer, reject}\}$. In a direct mechanism, the test results specify whether to reject or hire the applicant. Moreover, the recommendation to hire is employer-specific. The applicant reports his type $\mu \in [0,1]$ and in response the certifier charges the applicant a price $\varphi(\mu)$ and uses $\rho_\theta(\mu)$ to announce a recommendation $a \in A$.

A direct mechanism is *obedient* if it is optimal for each employer to offer employment if and only if recommended to do so. More precisely, obedience entails that the employers' beliefs are such that[14]

$$\mathbf{E}[\,\mu \mid a_t\,] \geq s_t > \mathbf{E}[\,\mu \mid a_b\,] \geq s_b > \mathbf{E}[\,\mu \mid r\,] \tag{1}$$

As test outcomes are public, obedience requires that the applicant follow the certifier's recommendation. The strict inequalities in (1) follow as an applicant (strictly) prefers the top employer over the bottom employer, and the bottom employer over rejection.
Let $\rho^t_\theta(\mu) := \rho_\theta(\mu)(a_t)$ and $\rho^b_\theta(\mu) := \rho_\theta(\mu)(a_b)$, these represent the probability of being hired by the top and bottom employers respectively conditional on ability θ for type μ's allocation. For a direct mechanism (ρ, φ), we can represent a test allocation as $\rho = (\rho^t_h, \rho^b_h, \rho^t_l, \rho^b_l)$.

A direct mechanism yields *gross utility* for type μ equal to $V(\mu) = \mu\left(\rho^t_h(\mu) + u\rho^b_h(\mu)\right) + (1 - \mu)\left(\rho^t_l(\mu) + u\rho^b_l(\mu)\right)$. Define the *indirect utility* function of a direct mechanism (ρ, φ) by

$$\mathcal{U}(\mu) := V(\mu) - \varphi(\mu)$$

When type μ misreports $\nu \neq \mu$ he earns gross utility given by

$$V(\mu, \nu) = \mu\left(\rho^t_h(\nu) + u\rho^b_h(\nu)\right) + (1 - \mu)\left(\rho^t_l(\nu) + u\rho^b_l(\nu)\right)$$

An direct mechanism is *incentive compatible* if for every μ, ν

$$\mathcal{U}(\mu) \geq V(\mu, \nu) - \varphi(\nu).$$

[13]Without the commitment assumption, the certifier can manipulate the applicant supply such that the expected ability of the applicant, conditional on being hired, is as low as possible $\left(= \frac{-v_l}{v_h - v_l}\right)$. Sequential rationality of the employer then requires the employer to hire the applicant.

[14]For an incentive compatible direct mechanism (φ, ρ), whenever $\int_0^1 \mu\rho_h(\mu)(a)\,dF(\mu) + \int_0^1 (1-\mu)\rho_l(\mu)(a)\,dF(\mu) > 0$ Bayes rule implies

$$\mathbf{E}[\mu|a] := \frac{\int_0^1 \mu\rho_h(\mu)(a)\,dF(\mu)}{\int_0^1 \mu\rho_h(\mu)(a)\,dF(\mu) + \int_0^1 (1-\mu)\rho_l(\mu)(a)\,dF(\mu)}$$

When $\int_0^1 \mu\rho_h(\mu)(a)\,dF(\mu) + \int_0^1 (1-\mu)\rho_l(\mu)(a)\,dF(\mu) = 0$, the action a is never recommended and can be removed from (1).

A direct mechanism is *individually rational* if $\mathcal{U}(\mu) \geq 0$ for all μ.

Given hiring standards (s_t, s_b), the intermediary seeks to maximize revenue generated by selling tests to applicants. By the revelation principle (Myerson (1986), Forges (1986)), it suffices to restrict attention to direct, obedient, and incentive compatible mechanisms (ρ, φ) with full participation. Thus, the certifier's objective is to maximize revenue

$$\Pi = \int_0^1 \varphi(\mu) dF(\mu)$$

among obedient, incentive compatible, individually rational direct mechanisms.
Moreover, by the definition of indirect utility for an obedient, incentive compatible, individually rational direct mechanism, we have the following

$$\Pi = \int_0^1 [\ V(\mu) - \mathcal{U}(\mu)\ ]\ dF(\mu)$$

Given such a mechanism with test allocation ρ, the payoff to the employers is

$$U_t(\rho\ ) := \int_0^1 [\ \mu\ v_h\ \rho_h^t(\mu) + (1-\mu)\ v_l\ \rho_l^t(\mu)\ ]\ dF(\mu)$$

$$U_b(\rho) := \int_0^1 [\ \mu\ v_h\ \rho_h^b(\mu) + (1-\mu)\ v_l\ \rho_l^b(\mu)\ ]\ dF(\mu)$$

When the choice of test allocation is unambiguous, I will drop dependence on ρ from the notation of employer payoffs.

4 No Screening Benchmark

I will first present the benchmark for the certifier's problem without screening frictions. In this section, the applicant can not misreport his type in a direct mechanism, and thus, all test allocations are incentive compatible. As the certifier faces no incentive compatibility restrictions, it can extract all surplus generated. This allocation is clearly individually rational. We are only left with obedience constraints, which constitute the certifier's information design problem. In particular, after observing the hiring standards (s_t, s_b), the optimal mechanism (ρ, φ) solves

$$\max_{\rho} \int_0^1 V(\mu) dF(\mu)$$

Such that ρ satisfies (1).
Thus, the design of optimal tests is as if the applicant can commit ex-ante to a costless type-dependent test. I refer to this information design problem as the efficient benchmark, and the distortions described in section 6 are relative to this benchmark. Here, the notion of efficiency is for the applicant-certifier market.

Proposition 1. *The set of equilibrium hiring standards (in pure strategy) is*

$$\mathcal{E} = \begin{cases} \left\{ \left(\frac{1}{u} \frac{-v_l}{v_h - v_l}, \frac{-v_l}{v_h - v_l} \right) \right\} & \text{if } u > \frac{-v_l}{v_h - v_l} \\ \{(s_t, s_b) = (1, x) \mid x \in [u, 1]\} & \text{otherwise} \end{cases}$$

Proof. See appendix A.1 □

Remark. In the absence of a competing employer, the top employer would always commit to accepting an applicant only if it is certain that the applicant's ability is $\theta = h$. In response, the certifier chooses a fully revealing test for all types. This gives the employer the highest possible surplus as $v_h > 0 > v_l$. When there is a competing employer, the top employer faces the risk of losing out on potentially high ability applicants to the bottom employer. Thus, competition restricts the top employer's selectivity.

The certifier maximizes the applicant's gross utility subject to obedience (1). Allocating an applicant to the top employer yields a higher utility, but it comes at the cost of pooling fewer low ability applicants. When the top employer sets a higher standard than the bottom employer, the certifier might find it beneficial to allocate applicants to the bottom employer. If s_b is sufficiently small, the lower marginal value of allocating to the bottom employer is overshadowed by the greater mass of applicants that the bottom employer accepts. The interplay between these forces determines what tests are offered. The obedience constraints must hold with equality as $s_t, s_b \geq \frac{-v_l}{v_h - v_l} > \mathbf{E}_F[\mu]$. If the top (bottom) employer hires with positive probability, then the expected ability of the hired applicant equals the standard s_t (s_b). If an employer's obedience constraint is slack, then the certifier can pool in more low ability applicants to that employer's recommendation, increasing the applicant's gross utility. The trade-off described above can be seen by substituting the obedience constraints in the certifier's objective.

$$\mathbf{E}\left[\frac{\mu}{s_t} \rho_h^t(\mu) + u \frac{\mu}{s_b} \rho_h^b(\mu) \right]$$

The terms in front of ρ_h^t and ρ_h^b represent the sum of the direct value from test allocation conditional on high ability and the indirect value from pooling additional low ability applicants. A simple pointwise maximization scheme then gives the solution. If $us_t < s_b$, the certifier only allocates applicants to the top employer. If $us_t > s_b$, the certifier only allocates the applicant to the bottom employer.

When the applicants' utility from joining the bottom employer is low enough, $u < \frac{-v_l}{v_h - v_l}$, the certifier always chooses to allocate the applicant to the top employer. Anticipating this, the top employer chooses $s_t = 1$.

If $\frac{-v_l}{v_h - v_l} < u$, then the top employer can not set $s_t = 1$. Setting a high threshold gives the bottom employer incentive to undercut the top employer by choosing $s_b = u(1 - \varepsilon)$. More generally, given some s_t the bottom employer can undercut the top employer whenever there exists some $\varepsilon > 0$ such that $u(1-\varepsilon)s_t > \frac{-v_l}{v_h - v_l}$. In equilibrium, the employers must not have an incentive to lower their standards to undercut the opponent. Moreover, if $us_t < s_b$, then the top employer has a profitable deviation $s_t = \frac{s_b}{u} - \varepsilon$ for some $\varepsilon > 0$. Thus, the employer must choose $s_b = us_t = \frac{-v_l}{v_h - v_l}$. Importantly, in equilibrium, the designer is indifferent between all the employers that hire with positive probability. By the same argument, only the top employer hires with positive probability in equilibrium.

Remark. The bottom employer is ***weak*** if the applicant's utility from being hired by the bottom employer is below the employer's reservation expected ability, $u < \frac{-v_l}{v_h - v_l}$. An important observation from Proposition 1 is that whenever the bottom employer is *weak*, the equilibrium certification mechanism and the top employer's standards are independent of the bottom employer's (sequentially rational) actions. This can be interpreted as a "no-entry" equilibrium, where the bottom employer is sufficiently undesirable (to the applicant) relative to the top employer. The top employer does not lower its standards, as the bottom employer is not able to poach applicants away from the top employer.

Preview of Results: I prove two main results that demonstrate how incentive constraints for the certifier's problem distort the (benchmark) allocation and its effect on employer competition.

In the benchmark allocation, the certifier allocates the applicant to the bottom employer with positive probability only if $us_t \geq s_b$. The first result (Theorem 1 in section 5.2) highlights the certifier's tendency to increase allocation to less selective firms as this reduces information rents (*informativeness-reducing* effect). Roughly speaking, Theorem 1 states that, under a regularity condition on the prior F, the certifier allocates the application to the bottom employer with positive probability even when $us_t < s_b$. Moreover, for $s_b - us_t$ sufficiently small (but positive), there is a reversal in the certifier's allocation relative to the benchmark; the certifier only allocates the applicant to the bottom employer.

The second result (Theorem 2 in section 6.2) demonstrates how the distortions from the screening frictions faced by the certifier affect the equilibrium determination of hiring standards and its effect on employer competition. In the benchmark allocation, *weak* bottom employer $\left(u < \frac{-v_l}{v_h - v_l} \right)$ does not affect the equilibrium certification mechanism and top employer's hiring standards. But due to the *informativeness-reducing* effect, the top employer is compelled to reduce its standards even against a *weak* bottom employer. More precisely, for uniformly distributed F, if the employers' reservation expected ability $\frac{-v_l}{v_h - v_l}$ is low enough, then there is no equilibrium in which the top employer sets the highest standard ($s_t = 1$) even against a weak bottom employer.

5 Certification Design with Privately Informed Applicant

I begin by describing how incentive compatibility affects the design of the certification mechanism. Consider some arbitrary but fixed hiring standard (s_t, s_b) such that $s_t > s_b \geq \frac{-v_l}{v_h - v_l}$.

Incentive Compatibility: Recall that a direct mechanism is incentive compatible if for every μ, ν

$$\mathcal{U}(\mu) \geq V(\mu, \nu) - \varphi(\nu).$$

An important quantity in the analysis of incentive compatibility is the *slope* of the indirect utility. Consider a function K defined by the difference in the applicant's utility conditional on $\theta = h$ and the applicant's expected utility conditional on $\theta = l$.

$$K(\mu) := \underbrace{\left(\rho_h^t(\mu) + u\rho_h^b(\mu)\right)}_{\text{utility conditional on } \theta = h} - \underbrace{\left(\rho_l^t(\mu) + u\rho_l^b(\mu)\right)}_{\text{utility conditional on } \theta = l}$$

As $V(\mu, \nu)$ is linear in the applicant's type μ, by Rochet (1987) and Chopra and Ely (2025), a test allocation ρ is incentive compatible for some price φ if and only the following monotonicity condition holds

$$K(\mu) = \left(\rho_h^t(\mu) + u\rho_h^b(\mu)\right) - \left(\rho_l^t(\mu) + u\rho_l^b(\mu)\right) \text{ is non-decreasing}$$

Moreover, the function K describes the slope of the applicant's indirect utility. In particular, $\mathcal{U}$ is convex and hence absolutely continuous. The indirect utility has the following integral representation

$$\mathcal{U}(\mu) = \mathcal{U}(0) + \int_0^\mu K(\nu)\, d\nu$$

Importantly, a test allocation ρ with lower variation in θ contingent outcomes results in a flatter indirect utility.

Obedience: Recall that the obedience constraint is given by

$$\mathbf{E}[\ \mu \mid a_t\ ] \geq s_t > \mathbf{E}[\ \mu \mid a_b\ ] \geq s_b > \mathbf{E}[\ \mu \mid r\ ]$$

Revenue maximization requires that the top employer's obedience constraint $\mathbf{E}[\ \mu \mid a_t\ ] \geq s_t$ holds with equality. Assume for contradiction that ρ generates constraint $\mathbf{E}[\ \mu \mid a_t\ ] > s_t$, then the certifier can improve its revenue by offering a lottery over test allocations. This lottery allocates all types with a small probability α, regardless of θ, to the top employer, and with probability $(1 - \alpha)$ the improving lottery allocates according to ρ. This preserves obedience and incentive compatibility, but generates a greater applicant gross utility without increasing information rents, thus increasing revenue. I present the argument for the binding bottom employer obedience constraint in appendix A.2. The argument follows by splitting a small mass of applicants allocated to the bottom employer, conditional on $\theta = h$, and allocating them between the top employer and rejection. Obedience constraints for a revenue maximizing mechanism can be expressed as follows[15]

$$\mathbf{E}[\ \mu \mid a_t\ ] = s_t > \mathbf{E}[\ \mu \mid a_b\ ] = s_b > \mathbf{E}[\ \mu \mid r\ ]$$

The equality of obedience constraints is similar to the informed certifier benchmark (section 4), but the argument used to perturb suboptimal mechanisms explicitly accounts for the application's incentive to misreport his type (incentive compatibility). From the definition of $\mathbf{E}[\mu|a]$, obedience constraint can be expressed by linear equalities, whenever $s_t > s_b \geq \mathbf{E}_F[\mu]$. [16]

$$(1 - s_t) \int_0^1 \mu \rho_h^t(\mu)\, dF(\mu) - s_t \int (1 - \mu) \rho_l^t(\mu)\, dF(\mu) = 0$$

$$(1 - s_b) \int_0^1 \mu \rho_h^b(\mu)\, dF(\mu) - s_b \int (1 - \mu) \rho_l^b(\mu)\, dF(\mu) = 0$$

Individual Rationality: Any incentive compatible and obedient direct mechanism is revenue maximizing only if the individual rationality constraint is binding for some type μ_0. More precisely, there exists a type μ_0 such that $\mathcal{U}(\mu_0) = 0$. Let μ_0 be the smallest type with binding individual rationality. The indirect utility is then given by

$$\mathcal{U}(\mu) = \begin{cases} -\int_{\mu}^{\mu_0} K(\nu) d\nu, & \text{if } \mu \leq \mu_0 \\ \int_{\mu_0}^{\mu} K(\nu) d\nu, & \text{if } \mu \geq \mu_0 \end{cases}$$

The revenue can then be expressed as the difference between gross and indirect utility

$$\Pi = \int_0^1 [\ V(\mu) - \mathcal{U}(\mu)\ ]\, dF(\mu)$$

$$= \int_0^{\mu_0} \left\{ V(\mu) + \int_{\mu}^{\mu_0} K(\nu) d\nu \right\} f(\mu) d\mu + \int_{\mu_0}^1 \left\{ V(\mu) - \int_{\mu_0}^{\mu} K(\nu) d\nu \right\} f(\mu) d\mu$$

As $\mathcal{U}(\mu_0) = 0$, the slope of the indirect utility of all types lower than μ_0 is negative. Thus, the expression above increases when K increases for types $\mu \leq \mu_0$. Moreover, for $\mu < \mu_0$ either $\rho_l^t(\mu) > \rho_h^t(\mu)$ or $\rho_l^b(\mu) > \rho_h^b(\mu)$. Marginally increasing ρ_h^t or ρ_h^b for types just below μ_0 preserves incentive compatibility, and increases the revenue. But increasing ρ_h^t is not always feasible as it

[15] The argument for binding obedience constraints is independent of whether the certifier announces test outcomes publicly or not. Additionally, if $s_t \leq s_b$ then in any revenue-maximizing mechanism a_b is recommended with zero probability. Thus, it is without loss to assume public test outcomes.

[16] The equalities imply that the expected ability of an applicant being recommended a_b is $s_b < s_t$, hence the applicant does not deviate. As $s_t, s_b \geq \mathbf{E}[\mu]$, the obedience constraint for recommendation r follows from the law of total expectation.

might lead to a decrease in ρ_h^b and a possible violation of the obedience constraint (1). This happens only if $\rho_h^t(\mu) + \rho_h^b(\mu) = 1$, which along with a negative slope implies $\rho_l^t(\mu) > 0$. Using this, Lemma 1 demonstrates how a simple perturbation can still be constructed to increase revenue.

Lemma 1. *Let (ρ, φ) be an incentive compatible, obedient, and individually rational direct mechanism with indirect utility $\mathcal{U}$. The mechanism (ρ, φ) is revenue maximizing only if $\mathcal{U}(0) = 0$.*

Proof. See appendix A.3 □

Lemma 1 implies that the revenue maximizing incentive compatible and obedient mechanism requires that the lowest type has a binding individual rationality constraint ($\mathcal{U}(0) = 0$). Importantly, this implies that the slope of the indirect utility is everywhere non-decreasing and positive.

Distortions Relative to the Benchmark: Using integration by parts, we can then express the certifier's revenue from an optimal incentive compatible and obedient mechanism in the familiar virtual surplus representation, see Myerson (1981).

$$\mathbf{E}\left[(\rho_h^t(\mu) + u\rho_h^b(\mu)) \underbrace{\left(\mu - \frac{1-F(\mu)}{f(\mu)}\right)}_{\text{high ability virtual value}}\right] + \mathbf{E}\left[(\rho_l^t(\mu) + u\rho_l^b(\mu)) \underbrace{\left(1 - \mu + \frac{1-F(\mu)}{f(\mu)}\right)}_{\text{low ability virtual value}}\right]$$

The above expression reveals an interesting property of the certifier's problem; under standard monotone hazard rate conditions, see Myerson (1981), the high ability virtual value is increasing and crosses 0 from below, while the low ability virtual value is positive and decreasing. From the definition of K, we can express the objective as follows[17]

$$\mathbf{E}\left[\rho_h^t(\mu) + u\rho_h^b(\mu)\right] - \mathbf{E}\left[\left(1 - \mu + \frac{1-F(\mu)}{f(\mu)}\right) K(\mu)\right]$$

Plugging in the obedience constraint, which holds with equality, we can simplify the expression for the applicant's gross utility. We can simplify the objective to the following

$$\underbrace{\mathbf{E}\left[\frac{\mu}{s_t}\rho_h^t(\mu) + u\frac{\mu}{s_b}\rho_h^b(\mu)\right]}_{\text{Information design tradeoff}} - \underbrace{\mathbf{E}\left[\left(\frac{1-F(\mu)}{f(\mu)}\right) K(\mu)\right]}_{\text{Distortion from screening}} \tag{2}$$

The first term in (2) captures the information design tradeoff, studied in section 4. The second term is a distortionary term due to the certifier screening the applicant's private information. Notice that the revenue generated by the optimal mechanism in the benchmark is greater than the revenue generated when the certifier has to screen the applicant.

[17] To keep the notation simple, I do not explicitly index K by the test choice ρ.

5.1 Certifier's Optimization Problem

Solving the certifier's primal problem involves aggregate linear constraints (in the form of obedience) and monotonicity constraints (for incentive compatibility). The formal optimization problem can be summarized as

$$\max_{\rho:\ [0,1]\to\Delta(A)\times\Delta(A)} \mathbf{E}\left[V(\mu)\right] - \mathbf{E}\left[\frac{1-F(\mu)}{f(\mu)}K(\mu)\right] \tag{P}$$

subject to

$$\begin{aligned}
&(1-s_t)\int_0^1 \mu\rho_h^t(\mu)\,dF(\mu) - s_t\int(1-\mu)\rho_l^t(\mu)\,dF(\mu) \geq 0 && \text{(Top Employer Obedience)}\\
&(1-s_b)\int_0^1 \mu\rho_h^b(\mu)\,dF(\mu) - s_b\int(1-\mu)\rho_l^b(\mu)\,dF(\mu) \geq 0 && \text{(Bottom Employer Obedience)}\\
&K \geq 0 && \text{(Individual Rationality)}\\
&K \quad \text{non -decreasing} && \text{(Incentive Compatibility)}
\end{aligned}$$

Relaxed Problem: Consider the relaxation of the primal problem (P) which does not involve the monotonicity constraint for K.

$$\max_{\rho:\ [0,1]\to\Delta(A)\times\Delta(A)} \mathbf{E}\left[V(\mu)\right] - \mathbf{E}\left[\frac{1-F(\mu)}{f(\mu)}K(\mu)\right] \tag{R}$$

subject to

$$\begin{aligned}
&(1-s_t)\int_0^1 \mu\rho_h^t(\mu)\,dF(\mu) - s_t\int(1-\mu)\rho_l^t(\mu)\,dF(\mu) \geq 0\\
&(1-s_b)\int_0^1 \mu\rho_h^b(\mu)\,dF(\mu) - s_b\int(1-\mu)\rho_l^b(\mu)\,dF(\mu) \geq 0\\
&K \geq 0
\end{aligned}$$

In general, the solution to the relaxed problem (R) is not incentive compatible. To simplify the analysis and in the interest of focusing on economic insights, I require the prior cdf F to obey the following regularity assumption.

Assumption 1. The weighted inverse hazard ratio $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing.[18]

Importantly, assumption 1 leads to the following key simplification:

[18] A CDF F satisfies assumption 1 if and only if there exists a non-increasing, positive function $g:[0,1]\to\mathbb{R}$ such that

$$\lim_{x\uparrow 1}\exp\left(-\int_0^x \frac{1}{(1-\nu)g(\nu)}d\nu\right)=0 \quad \text{and} \quad F(\mu)=1-\exp\left(-\int_0^\mu \frac{1}{(1-\nu)g(\nu)}d\nu\right)$$

Proposition 2. *If assumption 1 holds, then there is a solution to the relaxed problem* (R) *that is also a solution to the primal problem* (P)*. Moreover under assumption 1, any solution to the primal problem* (P) *is also a solution of the relaxed problem* (R)*.*

Proof. See appendix A.4 □

Remark. The proposition follows from noting that a solution to the relaxed problem (R) has non-decreasing slope K whenever assumption 1 holds. To illustrate this, I will provide a simplified, assuming that the optimal test takes finitely many values, sketch which captures the essence of the more general argument in the appendix A.4.

Proposition 2 proof sketch: Fix some test allocation ρ. Consider two intervals I_1 and I_2 such that $I_1 < I_2$ in the strong set order.[19] Moreover, assume that ρ is constant on I_1 and I_2. Let the slope K be constant on these intervals. Assume for contradiction that $K(I_1) > K(I_2)$. As $K \geq 0$, one of the following holds

1. either $\rho_l^t(I_1) + u\rho_l^b(I_1) < \rho_l^t(I_2) + u\rho_l^b(I_2)$
2. or, $\rho_h^t(I_2) + u\rho_h^b(I_2) < \rho_h^t(I_1) + u\rho_h^b(I_1)$

I will show that under assumption 1, condition 1) can never happen in an optimal mechanism. Proof for 2) is analogous (see appendix A.4). Consider subsets $I_1' \subset I_1$ and $I_2' \subset I_2$ such that

$$\int_{I_1'} (1-\mu)\, dF(\mu) = \int_{I_2'} (1-\mu)\, dF(\mu) > 0 \tag{MC}$$

The main step here involves marginally changing the test allocation for types in intervals I_1' and I_2'. We shift the test allocation for types in interval I_1', conditional on $\theta = l$, in the direction of the allocation in interval I_2'. Similarly, we shift allocation for types in I_2' in the direction of allocation in I_1'. The new test allocation is intended to flatten the indirect utility for lower types and increase its steepness for higher types.

Assumption 1 implies that it is possible to construct a test, resulting from obedience-preserving (and gross surplus preserving) shifts, with steeper indirect utility. As the inverse hazard rate is decreasing, by assumption 1, the new allocation is a profitable deviation for the certifier. Roughly

A necessary condition for assumption 1 is the following **Bounded Hazard Rate** property

$$\frac{1-F(\mu)}{f(\mu)} \geq 1-\mu$$

This follows from $\lim_{\mu \uparrow 1} \frac{1-F(\mu)}{(1-\mu)f(\mu)} = 1$ and the fact that $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing. In particular, the inverse hazard rate is bounded below by a decreasing linear function.

[19]Let Y and Y' be subsets of $\mathbb{R}$. Set Y' dominates Y in the strong set order ($Y' \geq_{SSO} Y$) if for any x' in Y' and x in Y, we have $\max\{x', x\}$ in Y' and $\min\{x', x\}$ in Y.

speaking, the certifier benefits from pooling low ability applicants into the allocation for lower types.

Formally, we can perturb ρ_l on the sets I_1' and I_2'. Define, for $j \in \{t, b\}$ and $\varepsilon > 0$, the following perturbations

$$\widetilde{\rho}_l^j(I_1') := (1-\varepsilon)\rho_l^j(I_1) + \varepsilon\rho_l^j(I_2) \quad \text{and} \quad \widetilde{\rho}_l^j(I_2') := (1-\varepsilon)\rho_l^j(I_2) + \varepsilon\rho_l^j(I_1)$$

For equation (2), the choice above leaves the information design part of the revenue unchanged. Thus, the change in revenue from the perturbation is given by the following

$$\varepsilon\left((\rho_l^t(I_2) + u\rho_l^b(I_2) - \rho_l^t(I_1) - u\rho_l^b(I_1))\int_{I_1'}\left(\frac{1-F(\mu)}{f(\mu)}\right)\,dF(\mu)\right) \\ + \varepsilon\left((\rho_l^t(I_1) + u\rho_l^b(I_1) - \rho_l^t(I_2) - u\rho_l^b(I_2))\int_{I_2'}\left(\frac{1-F(\mu)}{f(\mu)}\right)\,dF(\mu)\right)$$

As $\rho_l^t(I_1) + u\rho_l^b(I_1) < \rho_l^t(I_2) + u\rho_l^b(I_2)$, the change in revenue from the perturbation is positive if the following holds

$$\int_{I_1'}\left(\frac{1-F(\mu)}{f(\mu)}\right)\,dF(\mu) - \int_{I_2'}\left(\frac{1-F(\mu)}{f(\mu)}\right)\,dF(\mu) \geq 0$$

Dividing and multiplying the integrands in the left-hand side of the above inequality by $(1-\mu)$ yields

$$\int_{I_1'}\frac{1-F(\mu)}{(1-\mu)f(\mu)}(1-\mu)\,dF(\mu) - \int_{I_2'}\frac{1-F(\mu)}{(1-\mu)f(\mu)}(1-\mu)\,dF(\mu)$$

By assumption 1 and (MC) the above is bounded below by

$$\left[\min_{\mu\in I_1'}\left\{\frac{1-F(\mu)}{(1-\mu)f(\mu)}\right\} - \max_{\mu\in I_2'}\left\{\frac{1-F(\mu)}{(1-\mu)f(\mu)}\right\}\right] \times \int_{I_1'}(1-\mu)\,dF(\mu) \geq 0$$

The perturbation $\widetilde{\rho}$ is obedient, implying ρ is not optimal.

From the argument above (and Appendix A.4), any solution to the relaxed problem (R) can be rearranged to yield a payoff equivalent test allocation with non-decreasing slope K. The second statement of Proposition 2 follows from noting that a solution to the relaxed problem with non-decreasing slope is also a solution to the primal problem (P).

5.2 Increased Affinity to Less Selective Employers

The key theoretical insight here is that second-degree price discrimination by the certifier steers allocations towards less selective employers. Less selective employers hire applicants with lower expected ability, allowing the certifier to offer tests with lower payoff variation (heterogeneity) across applicant types. The lower variation in test outcomes lowers information rents by flattening the applicant's indirect utility. This highlights a unique wedge resulting from the interaction of screening and information design components of the certifiers' problem. The certifier distorts the test allocation by reducing the overall informativeness of tests.

In the benchmark (section 6.1), if $us_t < s_b$ then the certifier allocates the applicant only to the top employer. The first statement of Theorem 1 shows a reversal in test allocation when the certifier has to screen the applicant. For standards with $s_b - us_t$ sufficiently small (but positive), the certifier allocates the applicant only to the bottom employer. The second statement of Theorem 1 shows that for any choice of standards $s_t > s_b$, the certifier **never** chooses an allocation that allocates the applicant only to the top employer. Formally, we get the following result.

Theorem 1. *Under assumption 1, there exists a constant $\zeta > 0$ such that if $\zeta > s_b - us_t$ then the optimal incentive compatible, obedient and individually rational direct mechanism recommends a_t with zero probability ($\rho^t = 0$).*
Moreover, under assumption 1, if $s_t > s_b$ then the optimal incentive compatible, obedient and individually rational direct mechanism recommends a_b with positive probability ($\rho_h^b \neq 0$).

Proof. See appendix A.5 □

Remark. Theorem 1 highlights the underlying channel through which the screening frictions faced by the certifier shape employer competition. In the benchmark in section 4, the certifier only allocates the applicant to the bottom employer if $us_t > s_b$. When the bottom employer sets a lower standard than the top employer, the certifier can pool a greater mass of low ability applicants if it allocates the applicant to the bottom employer over the top employer. The distortions from the screening by the certifier amplify this effect. To observe this, recall the certifier's revenue is given by (2)

$$\mathbf{E}\left[\frac{\mu}{s_t}\rho_h^t(\mu) + u\frac{\mu}{s_b}\rho_h^b(\mu)\right] - \mathbf{E}\left[\frac{1-F(\mu)}{f(\mu)}K(\mu)\right]$$

The slope $K(\mu)$ is given by the difference in the applicant's utility conditional on $\theta = h$ and the applicant's utility conditional on $\theta = l$.

$$K(\mu) = \left(\rho_h^t(\mu) + u\rho_h^b(\mu)\right) - \left(\rho_l^t(\mu) + u\rho_l^b(\mu)\right)$$

A reduction in ability contingent variation (heterogeneity) of test outcomes decreases the ability contingent variation in the applicant's utility, resulting in lower information rents. A larger mass of low ability applicants reduces the difference in the test outcome distribution conditional on θ. Due to this additional benefit, the certifier "overallocates" the applicant to the less selective employer, even if it reduces the applicant's gross utility. The inefficiency in test allocation thus reduces the informativeness of the tests provided by the certifier.

6 Equilibrium Analysis

Having described the certifier's response to standards chosen by the employers, we are ready to describe the equilibrium. The feedback loop between the certification mechanism and hiring standards leads to exclusion and narrower hiring standards.

Equilibrium Well-posedness: In general, for any given pair of standards (s_t, s_b), there can be many optimal mechanisms. More precisely, the set of solutions to (P) might not be a singleton. Thus, the certifier's sequential rationality alone does not pin down the certification mechanism. This is particularly troublesome when the payoffs of the top and the bottom employers vary across different solutions.[20] Conveniently for any given (s_t, s_b), under assumption 1, the payoffs of the employers are constant over the set of optimal mechanisms. Moreover, the solution set of the relaxed program varies continuously with the choice of (s_t, s_b). This establishes that under assumption 1, an equilibrium of the certification game exists with employers playing possibly mixed strategies in the first period. Lemma 2 formilizes these claims

Lemma 2. *Let F satisfy assumption 1. For each $(s_t, s_b) \in \left[\frac{-v_l}{v_h - v_l}, 1\right]^2$, the employer payoffs U_t and U_b are constant over the set of solutions to the primal problem (P). Moreover, the payoffs U_t and U_b vary continuously with (s_t, s_b).*

Proof. See appendix A.7 □

Let $\sigma = (\sigma_t, \sigma_b) \in \Delta([0,1]) \times \Delta([0,1])$ represent a mixed strategy profile played by the employers.

6.1 Single Employer

To demonstrate the equilibrium interaction of hiring standards and certification mechanisms, I will consider a slight variation of the model with only the top employer, or equivalently $u = 0$. Given a standard $s_t > \mathbf{E}[\mu]$, the obedience constraint is

$$(1 - s_t)\mathbf{E}\left[\mu \rho_h^t(\mu)\right] - s_t \mathbf{E}\left[(1-\mu)\rho_l^t(\mu)\right] \geq 0$$

Writing $\rho_l^t = \rho_h^t + K$ and rearranging yields

$$\mathbf{E}\left[\left(1 - \frac{\mu}{s_t}\right)\rho_h^t(\mu)\right] \leq \mathbf{E}\left[(1-\mu)K(\mu)\right]$$

By equation 2, we get the certifier's objective

$$\mathbf{E}\left[\frac{\mu}{s_t}\rho_h^t(\mu)\right] - \mathbf{E}\left[\frac{1-F(\mu)}{f(\mu)}K(\mu)\right]$$

[20] In this case, the description of an equilibrium will involve the employer's conjectures about the certifier's response to a deviation from equilibrium hiring standards. This might be dependent on ad hoc tie-breaking assumptions.

Example: Assume that the distribution F is uniform, which implies $\frac{1-F(\mu)}{f(\mu)} = 1 - \mu$. The revenue expression simplifies further by plugging in the definition of K and using the binding obedience constraint to substitute $\mathbf{E}\left[\frac{1-F(\mu)}{f(\mu)} K(\mu)\right]$ with $\mathbf{E}\left[\left(1 - \frac{\mu}{s_t}\right) \rho_h^t(\mu)\right]$ and is given by

$$\mathbf{E}\left[\left(\frac{2\mu}{s_t} - 1\right) \rho_h^t(\mu)\right]$$

Pointwise maximizing the integrand gives that $\rho_h^i(\mu) = 1$ for all $\mu \geq s_t/2$ and 0 otherwise. The corresponding ρ_l^t is such that $\rho_l^t(\mu) = \alpha$ if $\mu \geq s_t/2$ and 0 otherwise, where α solves

$$\alpha\, s_t \int_{s_t/2}^{1} (1-\mu)\, d\mu \;=\; (1-s_t) \int_{s_t/2}^{1} \mu\, d\mu$$

The pointwise optimal ρ_h^t does not satisfy the obedience constraint for $s_t < 2/3$, as the corresponding $\alpha > 1$. To restore obedience, the optimal solution involves allocating types below $s_t/2$ to the top employer. When $s_t < 2/3$, the optimal mechanism is a bang-bang test. All types below a threshold $\mu_0 < s_t/2$ are excluded (allocated to the top employer with zero probability) and all types above μ_0 are allocated to the top employer with probability 1, independent of θ. Obedience constraint then implies that μ_0 solves

$$s_t \int_{\mu_0}^{1} (1-\mu)\, d\mu \;=\; (1-s_t) \int_{\mu_0}^{1} \mu\, d\mu$$

Recall the top employer's payoff from test allocation ρ is

$$U_t = \mathbf{E}\left[\mu\, v_h\, \rho_h^t(\mu) + (1-\mu)\, v_l\, \rho_l^t(\mu)\right]$$

By binding obedience constraint for hiring standard s_t

$$U_t = \left(v_h + v_l \frac{1-s_t}{s_t}\right) \mathbf{E}\left[\mu \rho_h^t(\mu)\right]$$

Given the designer's optimal response to $s_t \geq 2/3$, the top employer's payoff is

$$\left(v_h + v_l \frac{1-s_t}{s_t}\right) \int_{s_t/2}^{1} \mu\, d\mu$$

This expression is increasing in s_t whenever $-v_h \geq v_l$, this holds as $\frac{-v_l}{v_h - v_l} > 1/2$. Thus, the optimal hiring standard is $s_t = 1$. When $s_t < 2/3$ the allocation takes the bang-bang form mentioned before. As $1/2 = \mathbf{E}[\mu] < \frac{-v_l}{v_h - v_l} \leq s_t$, the threshold $\mu_0 < s_t/2 \leq 1/3$. Moreover, the binding obedience constraint implies that the designer responds to an increase in s_t by increasing threshold μ_0. As $\mu_0 < 1/3 < \frac{-v_l}{v_h - v_l}$, the expected ability of applicants that are excluded from the mechanism after marginally increasing s_t is below $\frac{-v_l}{v_h - v_l}$. Hence, when $s_t < 2/3$ the top employer has a profitable deviation by increasing it's hiring standards. Generalizing this example we have Proposition 3

Proposition 3. *Fix $u = 0$. If $\frac{1-F(\mu)}{f(\mu)}$ is convex and assumption 1 holds, then the top employer's payoff U_t is increasing in s_t. In particular, the equilibrium standard is such that $s_t = 1$ with probability* 1.

Proof. See appendix A.6 □

Remark. The above example illustrates how the applicant's private information distorts equilibrium hiring standards. Unlike the benchmark in section 4, the presence of private information leads to *exclusion*. To reduce information rents, the designer excludes lower types from the mechanism. Increasing standards results in more exclusion. Thus, the employer faces a tradeoff: higher standards improve the expected ability of each hire, but reduce the hiring probability. In the uniform example, this trade off is resolved in the top employer raising its standards.

Generally, the designer can respond to increased standards in one of two possible ways. First, increasing the probability with which the applicant gets hired. The expected ability of these new applicants must be greater than the employer's standard, and hence greater than $\frac{-v_l}{v_h - v_l}$. In particular, this leads to a greater payoff for the employer. Second, by decreasing the probability with which the applicant gets hired. The expected ability of the removed applicant types must be less than the employer's standard. But the expected ability of removed applicants could still be greater than $\frac{-v_l}{v_h - v_l}$, in which case the employer's payoff decreases. In the example, the designer chooses the latter of the two responses. Yet, the employer finds it profitable to increase standards as the expected ability of removed applicant types is lower than $\frac{-v_l}{v_h - v_l}$.

If $\frac{1-F(\mu)}{f(\mu)}$ is convex and $\frac{1-F(\mu)}{(1-\mu)f(\mu)}$ is non-increasing, then the designer responds by decreasing the mass of hired applicants. But the expected ability of these applicants is below $\mathbf{E}[\ \mu\ ]$. Hence, the employer sets $s_t = 1$.

6.2 Employer Competition

The choice of hiring standards represents how selective an employer can be; having more market power corresponds to a more selective employer. The *selectivity ratio* given by $\frac{s_b}{s_t}$ captures the bottom employer's selectivity relative to the top employer. A higher selectivity ratio is indicative of a more competitive employer market.

Combining Theorem 1 and Proposition 3, we get that there is no equilibrium in pure strategy for which $s_b/s_t \leq u$. Whereas in the benchmark, in section 4, the equilibrium involves $s_b/s_t = u$. This observation highlights the connection between the distortion in Theorem 1 and employer competition.[21] The increase in the bottom employer's selectivity relative to the top employer,

[21] The caveat being that the model does not always admit an equilibrium in pure strategies, and verifying the existence of a pure strategy equilibrium requires conditions on employer payoff that are endogenously determined by the choice of certification mechanism.

compared to the benchmark in section 4, is a consequence of the certifier's tendency to steer allocations in favor of less selective employers (Section 5.2).

Recall, in section 4, an bottom employer is *weak* whenever the applicant's utility from being hired by the bottom employer is below the employer's reservation standard, $\frac{-v_l}{v_h - v_l} > u$. In the benchmark, when the bottom employer is *weak*, the presence or absence of the bottom employer does not affect the equilibrium certification mechanism and the top employer's standard. Theorem 2 shows that even against a *weak* bottom employer, the top employer is compelled to lower its standards because of the certifier's "overallocation" to the bottom employer. This indicates increased competition among employers.

Theorem 2. *Let F be uniformly distributed on $[0,1]$. There exists constants $C_2 > C_1 > 1/2$ such that if $C_1 < u < \frac{-v_l}{v_h - v_l} < C_2$, then σ is part of an equilibrium only if $\sigma_t(1) = 0$, i.e. in equilibrium the top employer sets standard $s_t = 1$ with zero probability.*

Proof. See Appendix A.8 □

Theorem 2 shows a peculiar spillover effect that allocative inefficiencies have on employer competition. The theorem is driven by two central features of the certification market. First, the nature of inefficiencies resulting from the interaction of screening and information design (Theorem 1). Second, the hiring standards shape the certifier's information design problem, resulting in the equilibrium dependence of the certification mechanism and employers' hiring standards. Distortions from screening have an *informativeness-reducing* effect on test allocation, skewing the applicant supply to less selective employers. The bottom ("less desirable") employer can benefit from this effect by setting higher hiring standards, relative to the benchmark. This forces the top ("more desirable") employer to lower its standards to prevent the bottom employer from poaching away too many potentially high ability applicants.

7 Extensions

7.1 Wages

Consider the base model with the change that instead of setting hiring standards, now the employers compete over wages. The employers simultaneously announce wages. In response, the certifier designs a mechanism. The applicant is hired if his expected value to the employer, conditional on test outcomes, is greater than the wage offered. To model imperfect competition through vertical differentiation, I assume that w dollars from the top employer are worth $\gamma_t(w) > w$ to the applicant and w dollars from the bottom employer are worth w to the applicant. Like before, we can apply the revelation principle to focus on incentive compatible, individually rational, and obedient direct mechanisms.

Given wages $(w_t, w_b) \in [0, v_h]^2$, the obedience constraints for the employers are

$$\mathbf{E}[\ \mu v_h + (1-\mu) v_l \mid a_t\ ] \geq w_t$$

and

$$\mathbf{E}[\ \mu v_h + (1-\mu) v_l \mid a_b\ ] \geq w_b$$

Conditional on hiring, an employer's profit is the expected productivity of the applicant minus the wage.

The applicant's obedience requires that he join the employer that is recommended by the certifier. Type μ applicant's gross utility from reporting ν

$$V_w(\mu,\nu) = \mu \left(\gamma_t(w_t) \rho_h^t(\nu) + w_b \rho_h^b(\nu) \right) + (1-\mu) \left(\gamma_t(w_t) \rho_l^t(\nu) + w_b \rho_l^b(\nu) \right)$$

This reduces to the base model in section 3 by defining

$$s_t = \frac{w_t - v_l}{v_h - v_l}, \ \ s_b = \frac{w_b - v_l}{v_h - v_l}, \ \text{ and } \ u = \frac{w_b}{\gamma_t(w_t)}$$

We can express the applicant's gross utility as

$$V_w(\mu,\nu) = \gamma_t(w_t) \times V(\mu,\nu)$$

The results from section 5 can be applied to this setting. In particular, the obedience constraint for the employers is binding. The expected value of the applicant, conditional on hiring, equals the offered wage, leading to zero profit for employers. The certifier's ability to flexibly design information allows it to freely pool low ability applicants, reducing the employers' profits. In fact, the model predicts that vertical differentiation of employers alone does not lead to meaningful employer competition when the certifier can flexibly design and price information.

7.2 Two Sided Platform

Consider an extension of the base model in which the certifier can charge the employers for participating in the mechanism. This might be the case in two-sided platforms for matching workers with employers. To fix ideas, I model the fee as a fixed fraction of the employers' profits from the applicant match. Let $\alpha = (\alpha_t, \alpha_b) \in [0,1]$ represent the fraction of employers' surplus that the certifier charges the employers for using its services. Like before, we can restrict attention to incentive compatible, individually rational, and obedient direct mechanisms. The analysis of the applicant's and employers' problems is mostly unchanged. The certifier's revenue from a test allocation ρ is given by

$$\alpha_t U_t(\rho) + \alpha_b U_b(\rho) + \mathbf{E}\left[\ V(\mu) - \mathcal{U}(\mu)\ \right]$$

The above expression simplifies to

$$\mathbf{E}\left[\,\mu\left(\,x_h^t\rho_h^t(\mu)+x_h^e\rho_h^b(\mu)\,\right)+(1-\mu)\left(\,x_l^t\rho_l^t(\mu)+x_l^b\rho_l^b(\mu)\,\right)\,\right]-\mathbf{E}[\,\mathcal{U}(\mu)\,]$$

Where $x_\theta^t = 1+\alpha_t v_\theta$ and $t_\theta^b = u+\alpha_b v_\theta$.
From the expression above, we can see that Proposition 2 extends to this setting. When the certifier can extract surplus from the employers, the test allocations might be more informative. The fee charged to the employers aligns the certifier's preference with the employers'. The distortion from screening will stiff affect the equilibrium hiring standards, but the effect is now confounded by the certifier's incentive to extract surplus from the employers. Depending on the weights α_i and α_e, certification could be skewed in favor of either of the employers.

7.3 Beyond Binary Ability

Employers often value applicants for many employment-relevant characteristics. This becomes especially interesting if employers value different characteristics differently. Putting aside the question of distortion to employer competition, designing an optimal mechanism in this case is a multi-dimensional screening problem. This poses new challenges in the design of an optimal mechanism and the subsequent equilibrium analysis. A first step towards this should involve the special case when all employers have similar preferences for different characteristics of an applicant. I don't explore this possibility, but I conjecture that the link between competition and applicants' private information remains in this special case.

8 Conclusion

The paper identifies a novel channel between this reduction in information and competition between employers. Second-degree price discrimination by the certifier reduces the overall informativeness of tests, and this can lead to increased competition among the employers, relative to the benchmark, when the certifier can efficiently allocate tests. Therefore, interventions in the certifier-applicant market might have unexpected consequences for employer competition. The rich incentive structure of the certifier's problem and its feedback with the employers' actions is shared by other economically relevant examples, like rating agencies in credit markets or quality assurance by sellers. Moreover, equilibrium feedback between the decision makers' (employers') actions and the certification mechanism highlights potential regulatory challenges in information markets more generally (see Bergemann and Bonatti (2019) for a survey of markets for information).

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A Appendix

A.1 Benchmark

Proposition 1. The equilibrium hiring standard (in pure strategy) is

$$(s_t, s_b) = \begin{cases} \left(\frac{1}{u}\frac{-v_l}{v_h-v_l}, \frac{-v_l}{v_h-v_l}\right) & \text{if} \quad u > \frac{-v_l}{v_h-v_l} \\ (1, x) & \text{for } x \in [u, 1] \text{ otherwise} \end{cases}$$

Proof. Note that in any equilibrium, the employer accepts an applicant only if the applicant's expected value is positive. Thus, $\frac{-v_l}{v_h-v_l}$ is a lower bound for the equilibrium standard set by either employer. Moreover, in any equilibrium $s_t \geq s_b$, as otherwise the top employer has a profitable deviation to $s_t = s_b$. The designer's problem can be stated as

$$\max \mathbf{E}\left[\mu \left(\rho_h^t(\mu) + u\rho_h^b(\mu) \right) + (1 - \mu)\left(\rho_l^t(\mu) + u\rho_l^b(\mu) \right) \right]$$

subject to

$$\mathbf{E}[\mu \mid a_t] \geq s_t > \mathbf{E}[\mu \mid a_b] \geq s_b > \mathbf{E}[\mu \mid r]$$

As $s_t, s_b \geq \frac{-v_l}{v_h-v_l} > \mathbf{E}[\mu]$, the obedience constraints are binding for a revenue maximizing testing policy. If the obedience constraints are slack then the certifier can do better by allocating more low ability applicants to the employer with slack obedience constraint. Plugging the binding obedience constraints into the expression for the designer's revenue yields

$$\mathbf{E}\left[\frac{\mu}{s_t}\rho_h^t(\mu) + u\frac{\mu}{s_b}\rho_h^b(\mu) \right]$$

The optimal test allocation is then

$$(\rho_h^t, \rho_h^b) = \begin{cases} (1, 0) & & \text{if} \quad us_t < s_b \\ (0, 1) & & \text{if} \quad us_t > s_b \\ (\alpha, 1-\alpha) & \text{for some } \alpha \in [0,1] & \textit{otherwise} \end{cases}$$

Note under binding obedience constraint the employer's payoff can be expressed as:

$$U_t(\rho) = \left(v_h + v_l\frac{1-s_t}{s_t}\right)\mathbf{E}\left[\mu\rho_h^t(\mu) \right]$$

$$U_b(\rho) = \left(v_h + v_l\frac{1-s_b}{s_b}\right)\mathbf{E}\left[\mu\rho_h^b(\mu) \right]$$

If $s_t < \max(1, s_b/u)$ then the top employer has a profitable deviation by increasing standards. If $us_t > s_b$ and $us_t > \frac{-v_l}{v_h-v_l}$ then the bottom employer prefers to increase it standard to $s_b = us_t - \varepsilon$ for small enough $\varepsilon > 0$.

Finally, in equilibrium, the bottom employer should be unable to undercut the top employer. Thus, $us_t \leq \frac{-v_l}{v_h-v_l}$. Moreover, the equilibrium test allocation has $(\rho_h^t, \rho_h^b) = (1, 0)$ as otherwise the top employer can undercut the bottom employer. This establishes the Proposition. □

A.2 Obedience

Consider ρ such that $\mathbf{E}[\ \mu \mid a_t\ ] = s_t > \mathbf{E}[\ \mu \mid a_b\ ] > s_b$ and a_b is recommended with positive probability. There must be some positive probability set S of types such that $\rho_h^b(\mu) > 0$ for all $\mu \in S$. For any positive probability subset $\hat{S} \subset S$, the designer can perturb the test allocation as described below. For all $\mu \in \hat{S}$, conditional on $\theta = h$, the new test allocates the applicant to the top employer with probability $\rho_h^t(\mu) + u\rho_h^b(\mu)$ and rejects with probability $1 - \rho_h^t(\mu) - u\rho_h^b(\mu)$. Otherwise, the test allocation is unchanged. The new test neither changes the gross utility nor the indirect utility, yielding the same revenue as ρ. This results in a slack obedience constraint for the top employer. The entrance obedience is not violated as the perturbation can be made over arbitrary subsets $\hat{S}$. Combining this with the argument in the main text implies that ρ is not revenue maximizing. Thus, for optimal mechanisms, the obedience constraint holds with equality.

A.3 Individual Rationality

Lemma 1. Let (ρ, φ) be an incentive compatible, obedient, and individually rational direct mechanism with indirect utility $\mathcal{U}$. The mechanism (ρ, φ) is revenue maximizing only if $\mathcal{U}(0) = 0$.

Proof. Consider (ρ, φ) such that the binding type is $\mu_0 > 0$. From the arguments in the main text, we get that there is some non-increasing function $\gamma : [0,1] \to [0,1]$ and $\varepsilon > 0$ such that $\gamma(\mu_0) = 0$, $\gamma(\mu) > 0$ for some $\mu \in [\mu_0 - \varepsilon, \mu_0]$ and the following perturbed test allocation $\hat{\rho}$ is feasible and incentive compatible

$$(\hat{\rho}_h^t, \hat{\rho}_h^b) = \begin{cases} (\ \rho_h^t(\mu) + \gamma(\mu),\ \rho_h^b(\mu)\ ) & \text{if} \quad \mu_0 - \varepsilon \le \mu < \mu_0 \text{ and } \rho_h^t(\mu) + \rho_h^b(\mu) < 1 \\ (\ \rho_h^t(\mu),\ \rho_h^b(\mu)\ ) & \textit{otherwise} \end{cases}$$

$$(\hat{\rho}_l^t, \hat{\rho}_l^b) = \begin{cases} (\ \rho_l^t(\mu) - \gamma(\mu),\ \rho_l^b(\mu)\ ) & \text{if} \quad \mu_0 - \varepsilon \le \mu < \mu_0 \text{ and } \rho_h^t(\mu) + \rho_h^b(\mu) = 1 \\ (\ \rho_l^t(\mu),\ \rho_l^b(\mu)\ ) & \textit{otherwise} \end{cases}$$

Let $S, S' \subset [\mu_0 - \varepsilon, \mu_0]$ such that $\rho_h^t(\mu) + \rho_h^b(\mu) < 1$ on S and $\rho_h^t(\mu) + \rho_h^b(\mu) = 1$ on S'. The change in revenue from this perturbation is

$$\int_0^{\mu_0} \left\{ \hat{V}(\mu) - V(\mu) + \int_\mu^{\mu_0} [\ \hat{K}(\nu) - K(\nu)\ ]\, d\nu \right\} f(\mu) d\mu$$

$$= \int_0^{\mu_0 - \varepsilon} \left[\int_{\mu_0 - \varepsilon}^{\mu_0} \gamma(\nu) d\nu \right] f(\mu) d\mu + \int_{\mu_0 - \varepsilon}^{\mu_0} \left[\int_\mu^{\mu_0} \gamma(\nu) d\nu \right] f(\mu) d\mu + \int_S \mu \gamma(\mu)\, f(\mu) d\mu - \int_{S'} (1 - \mu) \gamma(\mu)\, f(\mu) d\mu$$

$$= F(\mu_0 - \varepsilon) \int_{\mu_0 - \varepsilon}^{\mu_0} \gamma(\mu) d\mu + \int_{\mu_0 - \varepsilon}^{\mu_0} [F(\mu) - F(\mu_0 - \varepsilon)]\, \gamma(\mu) d\mu + \int_S \mu \gamma(\mu)\, f(\mu) d\mu - \int_{S'} (1 - \mu) \gamma(\mu)\, f(\mu) d\mu$$

$$= \int_{\mu_0 - \varepsilon}^{\mu_0} \gamma(\mu)\ F(\mu)\ d\mu + \int_S \mu \gamma(\mu)\, f(\mu) d\mu - \int_{S'} (1 - \mu) \gamma(\mu)\, f(\mu) d\mu$$

$$\geq \int_{\mu_0-\varepsilon}^{\mu_0} \gamma(\mu)\ F(\mu)\ d\mu - \int_{\mu_0-\varepsilon}^{\mu_0} (1-\mu)\gamma(\mu)\ f(\mu) d\mu$$

Notice that the change in revenue is composed of positive terms except for the reduction in the applicant's surplus from a deceased employment probability of low ablity applicants in the set S'. When $\int_{\mu_0-\varepsilon}^{\mu_0} \gamma(\mu)\ F(\mu)\ d\mu + \int_S \mu\gamma(\mu)\ f(\mu)d\mu - \int_{S'}(1-\mu)\gamma(\mu)\ f(\mu)d\mu < 0$ we can further perturb the test by offering a lottery of test $\hat{\rho}$ and the test $(1,0,1,0)$. If ρ is obedient, then $\hat{\rho}$ has slack obedience constraint for the top employer. In particular, we can further perturb the allocation into $\rho' = (1-\alpha)\hat{\rho} + \alpha(1,0,1,0)$. Where $\alpha \in (0,1)$ is chosen such that

$$\alpha s_t \int_0^1 (1-\mu)\ dF(\mu) - (1-\alpha)s_t \int_{\mu_0-\varepsilon}^{\mu_0} (1-\mu)\gamma(\mu)\ f(\mu)d\mu = \alpha(1-s_t) \int_0^1 \mu\ dF(\mu)$$

Rearranging the terms, we get the following

$$\alpha\left(1 - \frac{\mathbf{E}[\mu]}{s_t}\right) = (1-\alpha)\int_{\mu_0-\varepsilon}^{\mu_0} (1-\mu)\gamma(\mu)\ f(\mu)d\mu$$

As ε can be made arbitrarily small, such an α exists in the interval $(0,1)$. By the choice of α, the perturbed test ρ' is obedient.

For $s_b \geq us_t$ we get that $\Pi(\rho) \leq \frac{\mathbf{E}[\mu]}{s_t}$, as $\frac{\mathbf{E}[\mu]}{s_t}$ is the revenue generated by the optimal mechanism from the benchmark in section 4. In particular, we get the following

$$\alpha(\ 1 - \Pi(\rho)\ ) \geq (1-\alpha)\int_{\mu_0-\varepsilon}^{\mu_0} (1-\mu)\gamma(\mu)\ f(\mu)d\mu \geq (1-\alpha)\ (\ \Pi(\rho) - \Pi(\hat{\rho})\ )$$

This implies

$$\Pi(\rho') - \Pi(\rho) > 0$$

Thus, we can choose some $\alpha \in (0,1)$ such that the test ρ' is incentive compatible, obedient, and yields a greater revenue than ρ.

Whenever $s_b < us_t$ the optimal mechanism only allocates the applicant to the bottom employer or to being unemployed (bottom employer-only allocation). If $s_b < us_t$ then the bottom employer-only allocation can produce at least as much gross surplus as any other allocation. Additionally, as $s_b < s_t$, the certifier can pool a greater mass of low-ability applicants. Thus, a bottom employer-only allocation minimizes the information rent by reducing $K(\mu) = (\rho_h^t(\mu) + u\rho_h^b(\mu)) - (\rho_h^t(\mu) + u\rho_h^b(\mu))$ while keeping the applicant's gross surplus fixed. In this case, the argument in text shows that $\mathcal{U}(0) = 0$ as for a bottom employer-only allocation negative slope $K(\mu) < 0$ implies that $\rho_h^t(\mu) + \rho_h^b(\mu) < 1$.

□

A.4 Solution to Relaxed Problem

Geometry of Optimal Tests: For the formal analysis, consider ρ as an element of the Hilbert space $L_2([0,1] \to \mathbb{R}^4, F)$ with the usual norm topology and Borel sigma algebra. Where $\rho =$

$(\rho_h^t,\ \rho_h^b,\ \rho_l^t,\ \rho_l^b)$. I maintain this assumption for the rest of the appendix.
For the relaxed problem, we can describe a test allocation ρ as a function from $[0,1]$ to the closed convex polytope $\mathcal{C}$. Where

$$\mathcal{C} = \left\{ x \in \mathbb{R}_+^4 \;\middle|\; \begin{array}{l} x_1 + x_2 \leq 1 \\ x_3 + x_4 \leq 1 \\ x_1 + ux_2 - x_3 - ux_4 \geq 0 \end{array} \right\}$$

Recall, the relaxed problem (R) is given by the following

$$\max_{\rho:\ [0,1]\to\Delta(A)\times\Delta(A)} \mathbf{E}\left[\mu(\ \rho_h^t(\mu) + u\rho_h^b(\mu)\ ) + (1-\mu)(\ \rho_l^t(\mu) + u\rho_l^b(\mu)\ )\right] - \mathbf{E}\left[\left(\frac{1-F(\mu)}{f(\mu)}\right) K(\mu)\right]$$

subject to

$$(1-s_t)\int_0^1 \mu\rho_h^t(\mu)\, dF(\mu) - s_t \int (1-\mu)\rho_l^t(\mu)\, dF(\mu) \geq 0$$
$$(1-s_b)\int_0^1 \mu\rho_h^b(\mu)\, dF(\mu) - s_b \int (1-\mu)\rho_l^b(\mu)\, dF(\mu) \geq 0$$
$$K \geq 0$$

Define the feasibility set

$$\mathcal{F} := \left\{ \rho \in L_2([0,1] \to [0,1]^4) \mid \rho_h^t(\mu) + \rho_h^b(\mu) \leq 1,\ \rho_l^t(\mu) + \rho_l^b(\mu) \leq 1, K(\mu;\rho) \geq 0\ \ \forall\ \mu \in [0,1] \right\}$$

Here $\rho = (\rho_h^t,\ \rho_h^b,\ \rho_l^t,\ \rho_l^b)$. The set $\mathcal{F}$ is bounded, convex, and norm-closed.[22] As L_2 is a reflexive Banach space, $\mathcal{F}$ is weakly closed and hence weakly compact.[23] The objective function is a continuous linear functional of ρ, thus a solution to the relaxed problem exists in $\mathcal{F}$. The obedience constraints satisfy the usual Robinson constraint quantification, see Robinson (1975) and ch 8 of Luenberger (1997).[24] Thus for each solution ρ^* of (R) there exists a multiplier $\lambda^* = (\lambda_t^*, \lambda_b^*) \in \mathbb{R}_+^2$ such that the corresponding Lagrangian $\mathcal{L}$ has a saddle point at (ρ^*, λ^*). More precisely, for all $\rho \in \mathcal{F}$ and $\lambda \in \mathbb{R}_+^2$, the following holds

$$\mathcal{L}(\rho, \lambda^*)\ \leq \mathcal{L}(\rho^*, \lambda^*)\ \leq\ \mathcal{L}(\rho^*, \lambda)$$

Where the Lagrangian is given by

$$\mathcal{L}(\rho, \lambda) := \int_0^1 \langle\ l(\mu, \lambda),\ \rho(\mu)\rangle\, dF(\mu)$$

[22]Where norm closedness folows from the fact that convergence in L_2 norm implies pointwise convergence almost everywhere over a subsequence.
[23]See chapter 2 of Bonnans and Shapiro (2013) for details.
[24]This amounts to having non-empty interior of the set of all obedient test allocations in $\mathcal{F}$.

Where $\langle \, , \, \rangle$ is the usual inner product in $\mathbb{R}^4$ and

$$l(\mu,\lambda) := \left(\mu - \frac{1-F(\mu)}{f(\mu)},\ u\left(\mu - \frac{1-F(\mu)}{f(\mu)}\right), 1-\mu+\frac{1-F(\mu)}{f(\mu)},\ u\left(1-\mu+\frac{1-F(\mu)}{f(\mu)}\right) \right)$$
$$+ \left(\mu\lambda_t(1-s_t),\ \mu\lambda_b(1-s_b),\ -(1-\mu)\lambda_t s_t, -(1-\mu)\lambda_b s_b\right)$$

By the saddle point property above, we get

$$\mathcal{L}(\rho^*,\lambda^*) = \max_{\rho\in\mathcal{F}} \int_0^1 \langle\, l(\mu,\lambda^*),\ \rho(\mu)\rangle \, dF(\mu)$$
$$= \int_0^1 \max_{\rho(\mu)\in\mathcal{C}} \langle\, l(\mu,\lambda^*),\ \rho(\mu)\rangle \, dF(\mu)$$

By linearity of inner product and the fact that linear functions optimized over closed and convex polytopes achieve optima at an extreme point, we get[25]

$$\rho^*(\mu) \in \mathrm{ex}(\mathcal{C}) \quad \text{for all } \ \mu$$

The set $\mathrm{ex}(\mathcal{C})$ can be enumerated as[26]

$$\mathrm{ex}(\mathcal{C}) = \left\{ \begin{array}{l} (0,0,0,0) \\ (0,1,0,1),\ (u,0,0,1),\ (0,1,u,0),\ (1,0,1,0) \\ (1,0,0,1),\ (0,1,0,0) \\ (1,0,0,0) \end{array} \right\}$$

In particular, the optimal test allocation ρ^* takes finitely many values. This establishes the key assumption made in the proof of Proposition 2 sketched in section 5.1 about the optimal test taking finitely many values.

A.4.1 Proof of Proposition 2

To prove Proposition 2 we can then extend the argument from the main text (section 5.1) to arbitrary ordered sets $S_1 < S_2$ and their subsets S_1', S_2' instead of intervals.
To complete the proof, I will present the calculation for case 2), not shown in the main text.

Fix some test allocation ρ. Consider two subsets of $[0,1]$, S_1 and S_2 such that $S_1 < S_2$ in the strong set order. Let the slope K be constant on these intervals. Assuming for contradiction that $K(S_1) > K(S_2)$ and that

$$\rho_h^t(S_2) + u\rho_h^b(S_2) < \rho_h^t(S_1) + u\rho_h^b(S_1)$$

Consider subsets $S_1' \subset S_1$ and $S_2' \subset S_2$ such that

$$\int_{S_1'} \mu \, dF(\mu) = \int_{S_2'} \mu \, dF(\mu) > 0 \tag{MC2}$$

[25]For a closed convex polytope X the set $\mathrm{ex}(X)$ is the set of all extreme points of X.
[26]Note, the representation of $\mathrm{ex}(\mathcal{C})$ is to emphasis monotonicity of K.

We can perturb ρ_h on the sets S'_1 and S'_2. Define, for $j \in \{t, b\}$ and $\varepsilon > 0$, the following perturbations

$$\widetilde{\rho}^j_h(S'_1) := (1-\varepsilon)\rho^j_h(S_1) + \varepsilon\rho^j_h(S_2) \quad \text{and} \quad \widetilde{\rho}^j_h(S'_2) := (1-\varepsilon)\rho^j_h(S_2) + \varepsilon\rho^j_h(S_1)$$

From equation (2) we observe that the choice above leaves the information design part of the revenue unchanged. Thus, the change in revenue from the perturbation is positive whenever

$$\left(\rho^t_h(S_1) + u\rho^b_h(S_1) - \rho^t_h(S_2) - u\rho^b_h(S_2)\right)\left(\int_{S'_1}\left(\frac{1-F(\mu)}{f(\mu)}\right)dF(\mu) - \int_{S'_2}\left(\frac{1-F(\mu)}{f(\mu)}\right)dF(\mu)\right) \geq 0$$

As $\rho^t_h(S_2) + u\rho^b_h(S_2) < \rho^t_h(S_1) + u\rho^b_h(S_1)$, the change in revenue due to the perturbation is positive iff

$$\int_{S'_1}\left(\frac{1-F(\mu)}{f(\mu)}\right)dF(\mu) - \int_{S'_2}\left(\frac{1-F(\mu)}{f(\mu)}\right)dF(\mu) \geq 0$$

Dividing and multiplying the integrands by μ yields

$$\int_{S'_1}\frac{1-F(\mu)}{\mu f(\mu)}\mu\, dF(\mu) - \int_{S'_2}\frac{1-F(\mu)}{\mu f(\mu)}\mu\, dF(\mu)$$

By assumption 1 and (MC2) the above is bounded below by

$$\left[\min_{\mu\in S'_1}\left\{\frac{1-F(\mu)}{\mu f(\mu)}\right\} - \max_{\mu\in S'_2}\left\{\frac{1-F(\mu)}{\mu f(\mu)}\right\}\right] \times \int_{S'_1}\mu\, dF(\mu) \geq 0$$

This establishes that ρ can be perturbed into a test $\widetilde{\rho}$ which preserves obedience and has a non-decreasing slope K. Importantly, the last inequality holds strictly for all smooth cdf F with support $[0,1]$ as the function $\frac{1-F(\mu)}{f(\mu)}$ is non-increasing and $\frac{1}{\mu}$ is strictly decreasing.

A.5 Increased Affinity to Less Selective Employer

Theorem 1: If assumption 1 holds then there exists a constant $\zeta > 0$ such that if $\zeta > s_b - us_t$ then the optimal mechanism recommends a_t with zero probability ($\rho^t = 0$).
Moreover, under assumption 1, if $s_t > s_b$ then the optimal incentive compatible, obedient and individually rational direct mechanism recommends a_b with positive probability ($\rho^b_h \neq 0$).

Proof. Recall the Lagrangian for the relaxed problem (R) is given by $\mathcal{L}(\rho,\lambda) = \mathbf{E}[\ \langle l(\mu,\lambda),\rho(\mu)\rangle\ ]$ (see sction A.4) where

$$l(\mu,\lambda) = \left(\mu - \frac{1-F(\mu)}{f(\mu)},\ u\left(\mu - \frac{1-F(\mu)}{f(\mu)}\right), 1-\mu+\frac{1-F(\mu)}{f(\mu)},\ u\left(1-\mu+\frac{1-F(\mu)}{f(\mu)}\right)\right)$$
$$+\ (\mu\lambda_t(1-s_t),\ \mu\lambda_b(1-s_b),\ -(1-\mu)\lambda_t s_t, -(1-\mu)\lambda_b s_b)$$

For any optimal mechanism ρ^* and corresponding multiplier λ^* we describe the optimal test choice by it's epigraph.[27]

$$\text{epi}\langle\, l(\cdot,\lambda^*),\, \rho^*(\cdot)\rangle \;=\; \bigcap_{x\in \text{ex}(\mathcal{C})} \text{epi}\langle\, l(\cdot,\lambda^*),\, x\rangle$$

The equality follows from the saddle point property. The following conditions are sufficient for a test ρ^* to only assign the applicant to the bottom employer

$$\text{epi}\langle\, l(\cdot,\lambda^*),\, (0,1,0,0)\rangle \subseteq \text{epi}\langle\, l(\cdot,\lambda^*),\, (1,0,0,0)\rangle$$

$$\text{epi}\langle\, l(\cdot,\lambda^*),\, (0,0,0,1)\rangle \subseteq \text{epi}\langle\, l(\cdot,\lambda^*),\, (0,0,u,0)\rangle$$

$$\text{epi}\langle\, l(\cdot,\lambda^*),\, (0,1,0,0)\rangle \subseteq \text{epi}\langle\, l(\cdot,\lambda^*),\, (u,0,0,0)\rangle$$

$$\text{epi}\langle\, l(\cdot,\lambda^*),\, (0,1,0,1)\rangle \subseteq \text{epi}\langle\, l(\cdot,\lambda^*),\, (1,0,1,0)\rangle$$

The above holds simultaneously if the following equations hold simultaneously

$$\lambda_b^*(1-s_b) - \lambda_t^*(1-s_t) \geq 1-u,$$

$$u\lambda_t^* s_t - \lambda_b^* s_b \geq 0,$$

$$\lambda_b^*(1-s_b) - u\lambda_t^*(1-s_t) \geq 0,$$

$$\lambda_b^*(\mu-s_b) - \lambda_t^*(\mu-s_t) \geq 1-u$$

Let $s_t > s_b$, and set $\lambda_t^* = \lambda_b^* \frac{s_b}{us_t} + \varepsilon$ for small $\varepsilon > 0$.[28] By the choice of λ_t^* and $s_t > s_b$, the second and the third equations above are satisfied. Moreover, the first equation and the fourth hold if the following holds for all μ

$$\lambda_b^*\left(\mu - s_b - (\mu - s_t)\frac{s_b}{us_t}\right) - \varepsilon(\mu - s_t) \geq 1-u$$

Simplifying gives us the following sufficient condition

$$\lambda_b^* \frac{\mu(us_t - s_b) + (1-u)s_t s_b}{us_t} - \varepsilon(1-s_t) \geq 1-u$$

As ρ^* is a bottom employer-only allocation, it comprises of tests of the form $(0,0,0,0)$, $(0,1,0,1)$, or $(0,1,0,0)$. By monotonicity of K, the allocation $(0,1,0,0)$ is for higher types than test allocation of $(0,0,0,0)$ or $(0,1,0,1)$. Note that $\langle\, l(\cdot,\lambda),\, (0,0,0,0)\,\rangle = 0$ and $\langle\, l(\mu,\lambda),\, (0,1,0,1)\,\rangle = u\mu +$

[27]The epigraph a function $f : X \to [-\infty,\infty]$ valued in the extended real numbers is the set $\text{epi} f = \{(x,r) \in X \times \mathbb{R} : r \geq f(x)\}$

[28]If $s_t \leq s_b$ then the optimal allocation only allocates to the top employer with positive probability. In particular, this implies that $\zeta < (1-u)\frac{-v_l}{v_h - v_l}$.

$\lambda_b(\mu - s_b)$. In particular, $\langle\, l(\mu,\lambda),\ (0,1,0,1)\rangle$ crosses $\langle\, l(\cdot,\lambda),\ (0,0,0,0)\,\rangle$ at most once from below. We get that the optimal test is of the following form

$$(\ \rho_h^{t\,*}(\mu),\ \rho_h^{b*}(\mu),\ \rho_l^{t*}(\mu),\ \rho_l^{b*}(\mu)\ ) = \begin{cases} (0,0,0,0) & \mu \in [0,\mu_0) \\ (0,1,0,1) & \mu \in [\mu_0,\mu_1) \\ (0,1,0,0) & \mu \in [\mu_1, 1] \end{cases} \tag{ρ_{BO}}$$

As $\mathbf{E}[\mu] < \frac{-v_l}{v_h - v_l} \leq s_b < 1$, we have $0 < \mu_0 < 1$ and $\mu_0 < \mu_1 \leq 1$. By first order necessary conditions for μ_0, we get

$$\lambda_b^* = \frac{u}{s_b - \mu_0}$$

Positivity of the multipliers implies $\mu_0 < s_b$. Plugging this into the required inequality, we get

$$\frac{\mu(us_t - s_b) + (1-u)s_t s_b}{s_t(s_b - \mu_0)} - \varepsilon(1 - s_t) \geq 1 - u$$

As $\varepsilon > 0$ can be arbitrarily small, it suffices to consider

$$\frac{\mu(us_t - s_b) + (1-u)s_t s_b}{s_t(s_b - \mu_0)} > 1 - u$$

Dividing and multiplying by $s_t(s_b - \mu_0)$ and then simplifying the above inequality gives us the following sufficient condition

$$s_t\mu_0(1-u) > s_b - us_t$$

Note that the above holds whenever $s_b < us_t$.

We have shown that when the above condition holds, then there exists $\lambda^* \geq \mathbb{R}_+^2$ such that the bottom employer-only allocation (ρ_{BO}) optimizes the Lagrangian pointwise. In particular, (ρ_{BO}) is a solution to (R). The first statement of Theorem 1 follows from setting $\zeta = \frac{-v_l}{v_h - v_l}\frac{\mu_0^*(1-u)}{2}$. Here μ_0^* is the smallest value for μ_0 across all s_b. This is positive as $s_b \geq \frac{-v_l}{v_h - v_l} > \mathbf{E}[\mu]$. I present more details about this lower bound in section A.6. The important detail being $\mu_0^* > 0$ and hence $\zeta > 0$.

To prove the second statement, assume for contradiction that for some $s_t > s_b$ the optimal test allocation is top employer-only. By reasoning similarly to the proof of the first statement, the optimal top employer-only test is given as following

$$(\ \rho_h^{t\,*}(\mu),\ \rho_h^{b*}(\mu),\ \rho_l^{t*}(\mu),\ \rho_l^{b*}(\mu)\ ) = \begin{cases} (0,0,0,0) & \mu \in [0,\mu_0) \\ (1,0,1,0) & \mu \in [\mu_0,\mu_1) \\ (1,0,0,0) & \mu \in [\mu_1, 1] \end{cases} \tag{ρ_{TO}}$$

Where (ρ_{TO}) pointwise maximizes the Lagrangian $\mathcal{L}(\rho,\lambda) = \mathbf{E}[\ \langle l(\mu,\lambda), \rho(\mu)\rangle\ ]$ for some choice of multiplier $\lambda^* = (\lambda_t^*, \lambda_b^*)$ where

$$\lambda_t^* = \frac{1}{s_t - \mu_0}$$

and $s_t > \mu_0 > 0$. To establish a contradiction, consider the following two cases:
Case I: If $s_b - \frac{u}{\lambda_b^*} < \mu_0$ then the line $\langle\ l(\cdot, \lambda^*),\ (0,1,0,1)\ \rangle$ crosses $\langle\ l(\cdot, \lambda^*),\ (0,0,0,0)\ \rangle$ before $\langle\ l(\cdot, \lambda^*),\ (1,0,1,0)\ \rangle$ (i.e. before μ_0). Thus (ρ_{TO}) does not pointwise maximize $\mathcal{L}(\cdot, \lambda^*)$.
Case II: If $s_b - \frac{u}{\lambda_b^*} \geq \mu_0$ then the line $\langle\ l(\cdot, \lambda^*),\ (0,1,u,0)\ \rangle$ crosses $\langle\ l(\cdot, \lambda^*),\ (0,0,0,0)\ \rangle$ before $\langle\ l(\cdot, \lambda^*),\ (1,0,1,0)\ \rangle$ (i.e. before μ_0). To see this, let μ' be such that $\langle\ l(\mu', \lambda^*),\ (0,1,u,0)\ \rangle = 0$. In particular

$$u + \mu'(1 - s_b)\lambda_b^* - us_t(1 - \mu')\lambda_t^* = 0$$

Eearranging gives the following

$$\mu' \times (\ (1 - s_b)\lambda_b^* + us_t\lambda_t^*) = u\frac{\mu_0}{s_t - \mu_0}$$

As $s_b - \frac{u}{\lambda_b^*} \geq \mu_0$ and as $\lambda_t^* = \frac{1}{s_t - \mu_0}$ we get the following

$$\mu' \times \left(\frac{1 - s_b}{s_b - \mu_0} + \frac{s_t}{s_t - \mu_0} \right) \leq \frac{\mu_0}{s_t - \mu_0}$$

As $s_t > s_b$ we get that $\frac{1-s_b}{s_b-\mu_0} < \frac{1-s_t}{s_t-\mu_0}$, plugging this in the left-hand side above shows that

$$\mu' < \mu_0$$

Thus (ρ_{TO}) does not pointwise maximize $\mathcal{L}(\cdot, \lambda^*)$ for any value of $\lambda_b^* \in \mathbb{R}_+$. This establishes the second statement of the theorem. □

A.6 Single employer

Proposition 3: Fix $u = 0$. If $\frac{1-F(\mu)}{f(\mu)}$ is convex and assumption 1 holds, then the top employer's payoff U_t is increasing in s_t. In particular, the equilibrium standard is such that $s_t = 1$.

Proof. First, note that the employer never chooses $s_t < \mathbf{E}[\ \mu\ ] \leq \frac{-v_l}{v_h - v_l}$. Fix some $s_t \in (\frac{-v_l}{v_h - v_l}, 1)$. Let $T(s_t) = \mathbf{E}[\ V(\mu)\ ]$ be the expected gross utility generated by the optimal mechanism corresponding to standard s_t. As the obedience constraint must bind for the optimal mechanism, the employer's payoff is given by:

$$(\ s_t v_h + (1 - s_t)v_l\ )\ T(s_t)$$

The employer prefers a standard $s_t' > s_t$ over s_t if and only if

$$(\ s_t' v_h + (1 - s_t')v_l\ )\ T(s_t') - (\ s_t v_h + (1 - s_t)v_l\ )\ T(s_t) > 0$$

$$\iff\ s_t' T(s_t') - s_t T(s_t)\ >\ \frac{-v_l}{v_h - v_l}\ (\ T(s_t') - T(s_t)\ ) \qquad (3)$$

As argued in section A.5, under assumption 1, the optimal mechanism can be found among the ones with he following structure:

$$(\rho_h^t(\mu),\ \rho_l^t(\mu)) = \begin{cases} (0,0) & \mu \in [0,\mu_0) \\ (1,1) & \mu \in [\mu_0,\mu_1) \\ (1,0) & \mu \in [\mu_1,1] \end{cases}$$

Where $\int_{\mu_0}^{1} \mu dF(\mu) = \frac{s_t}{1-s_t} \int_{\mu_1}^{1} (1-\mu) dF(\mu)$.

For a given s_t, the optimal mechanism can be described by the tuple $(\mu_0(s_t),\ \mu_1(s_t))$. Consider the Lagrangian, $\mathcal{L}(\mu_0,\mu_1,\lambda;s_t)$, corresponding to the designer's constrained optimization problem, given by

$$\int_{\mu_0}^{\mu_1} (1+\lambda(\mu - s_t))\ dF(\mu) + \int_{\mu_1}^{1} \left(\mu(1+\lambda(1-s_t)) - \frac{1-F(\mu)}{f(\mu)} \right)\ dF(\mu)$$

Let $\mu_0(s_t),\ \mu_1(s_t),\ \lambda(s_t)$ be the optimal solutions. Note that $\lambda(s_t) \geq 0$ and by assumption 1 the inverse hazard rate $\frac{1-F(\mu)}{f(\mu)}$ is decreasing.

First, we will show that $\mu_0(s_t)$ is increasing. Using the first order necessary conditions with respect to μ_0 and μ_1 we get the following

$$\lambda(s_t) = \frac{1}{s_t - \mu_0(s_t)}$$

$$\lambda(s_t) = \frac{1}{s_t} \left[1 + \frac{1-F(\mu_1(s_t))}{(1-\mu_1(s_t))f(\mu_1(s_t))} \right]$$

First, I will handle the case when there is a corner solution with $\mu_1(s_t) = 1$. Only the first order necessary condition for μ_0 holds. By positivity of the multiplier, we get $\mu_0(s_t) < s_t$. By the saddle point property of the Lagrangian, we must have $1+\lambda(\mu - s_t) \geq \mu(1+\lambda(1-s_t)) - \frac{1-F(\mu)}{f(\mu)}$ for all $\mu \in [0,1]$. Importantly, $1+\lambda(\mu - s_t)$ becomes positive before $\mu(1+\lambda(1-s_t)) - \frac{1-F(\mu)}{f(\mu)}$ turns positive. As $1+\lambda(\mu-s_t)$ is positive for $\mu > \mu_0$ and as $\mu(1+\lambda(1-s_t)) - \frac{1-F(\mu)}{f(\mu)} \geq \mu - \frac{1-F(\mu)}{f(\mu)}$, we get that μ_0 is smaller than μ' where $\mu' - \frac{1-F(\mu')}{f(\mu')} = 0$. I will show later that this implies $\mu_0 < \mathbf{E}_F[\mu]$.

If the top employer marginally increases its standard from s_t to $s_t' > s_t$ and $\mu_1(s_t') = 1$. For obedience to hold $\mu_0(s_t') > \mu_0(s_t)$. The new mechanism removes types with the lowest expected ability, which is below $\mathbf{E}_F[\mu] < \frac{-v_l}{v_h - v_l}$, thus the top employer's payoff increases.

Now, let's consider the case with an interior solution. Define the following function of (μ_0,μ_1,s_t)

$$Q_1(\mu_0,\mu_1;s_t) := \frac{s_t - \mu_0}{s_t} \left[1 + \frac{1-F(\mu_1)}{(1-\mu_1)f(\mu_1)} \right] - 1$$

Now, consider the obedience constraint

$$(\ 1 - s_t)\ \int_{\mu_0(s_t)}^{1} \mu\ dF(\mu) - s_t \int_{\mu_0(s_t)}^{\mu_1(s_t)} (1-\mu)\ dF(\mu) = 0$$

Define the following function of (μ_0, μ_1, s_t)

$$Q_2(\mu_0, \mu_1; s_t) := (\ 1 - s_t)\ \int_{\mu_0}^{1} \mu\ dF(\mu) - s_t\ \int_{\mu_0}^{\mu_1} (1-\mu)\ dF(\mu)$$

Let $Q(\mu_0, \mu_1; s_t) := (\ Q_1,\ Q_2\ )$. By the first order condition and binding obedience constraint, we get

$$Q(\mu_0(s_t), \mu_1(s_t); s_t) = (0,0)$$

The Jacobian of Q is given by

$$J(\mu_0, \mu_1; s_t) := \begin{bmatrix} \frac{\partial}{\partial s_t} Q_1 & | & \frac{\partial}{\partial \mu_0} Q_1 & \frac{\partial}{\partial \mu_1} Q_1 \\ \frac{\partial}{\partial s_t} Q_2 & | & \frac{\partial}{\partial \mu_0} Q_2 & \frac{\partial}{\partial \mu_1} Q_2 \end{bmatrix} = \begin{bmatrix} J_s & | & J_\mu \end{bmatrix}$$

Where

$$J_s = \begin{bmatrix} \frac{\mu_0}{s_t^2} \left[1 + \frac{1-F(\mu_1)}{(1-\mu_1)f(\mu_1)} \right] \\ -\int_{\mu_0}^{1} \mu\ dF(\mu) - \int_{\mu_0}^{\mu_1} (1-\mu)\ dF(\mu) \end{bmatrix}$$

and

$$J_\mu = \begin{bmatrix} -\frac{1}{s_t} \left[1 + \frac{1-F(\mu_1)}{(1-\mu_1)f(\mu_1)} \right] & \frac{s_t - \mu_0}{s_t} \frac{\partial}{\partial \mu_1} \left(\frac{1-F(\mu_1)}{(1-\mu_1)f(\mu_1)} \right) \\ (s_t - \mu_0) f(\mu_0) & -s_t(1-\mu_1)f(\mu_1) \end{bmatrix}$$

From the above, we get

$$\det\ J_\mu = (1-\mu_1)f(\mu_1) \left[1 + \frac{1-F(\mu_1)}{(1-\mu_1)f(\mu_1)} \right]\ -\ \frac{(s_t - \mu_0)^2}{s_t} f(\mu_0) \frac{\partial}{\partial \mu_1} \left(\frac{1-F(\mu_1)}{(1-\mu_1)f(\mu_1)} \right)$$

Assumption 1 implies $\frac{\partial}{\partial \mu} \left(\frac{1-F(\mu)}{(1-\mu)f(\mu)} \right) \leq 0$, thus

$$\det\ J_\mu \mid_{(\mu_0(s_t), \mu_1(s_t), s_t)} > 0$$

By Implicit Function Theorem $(\ \mu_0(s_t),\ \mu_1(s_t)\ )$ can be described by an implicit function g. In particular $g(s_t) = (\ \mu_0(s_t),\ \mu_1(s_t)\ )$ and

$$\frac{\partial}{\partial\, s_t} g\ =\ -J_\mu^{-1}\ \times\ J_s$$

Where

$$J_\mu^{-1}\ = \frac{1}{\det\ J_\mu} \begin{bmatrix} -s_t(1-\mu_1)f(\mu_1) & -\frac{s_t - \mu_0}{s_t} \frac{\partial}{\partial \mu_1} \left(\frac{1-F(\mu_1)}{(1-\mu_1)f(\mu_1)} \right) \\ -(s_t - \mu_0) f(\mu_0) & -\frac{1}{s_t} \left[1 + \frac{1-F(\mu_1)}{(1-\mu_1)f(\mu_1)} \right] \end{bmatrix}$$

We can conclude that $\frac{\partial}{\partial s_t} \mu_0(s_t) \geq 0$ as

$$\frac{\mu_0(s_t)}{s_t}\ (1 - \mu_1(s_t))\ f(\mu_1(s_t))\ \left[1 + \frac{1 - F(\mu_1(s_t))}{(1-\mu_1(s_t))f(\mu_1(s_t))} \right] \\ - T(s_t)\ \frac{s_t - \mu_0(s_t)}{s_t}\ \frac{\partial}{\partial \mu_1}\ \left(\frac{1 - F(\mu_1(s_t))}{(1-\mu_1(s_t))f(\mu_1(s_t))} \right) > 0$$

This establishes that $\mu_0(s_t)$ is increasing.

Next, we will show that $\mu_1(s_t)$ is decreasing. After some algebra and using the fact that the obedience constraint binds, we get $\frac{\partial}{\partial s_t}\mu_1(s_t)$ equals

$$\frac{1}{\det J_\mu}\left[\frac{1}{s_t^2}\left(1+\frac{1-F(\mu_1(s_t))}{(1-\mu_1(s_t))f(\mu_1(s_t))}\right)\left(\mu_0(s_t)(s_t-\mu_0(s_t))f(\mu_0(s_t))-\int_{\mu_0(s_t)}^{1}\mu\, dF(\mu)\right)\right]$$

The sign of $\frac{\partial}{\partial s_t}\mu_1(s_t)$ is determined by

$$\mu_0(s_t)(s_t-\mu_0(s_t))f(\mu_0(s_t))-\int_{\mu_0(s_t)}^{1}\mu\, dF(\mu)$$

$$<\mu_0(s_t)\ \times\ (\ (1-\mu_0(s_t))f(\mu_0(s_t))-1+F(\mu_0(s_t))\ )$$

Assumption 1 implies the bounded hazard rate property. Thus, we get

$$(1-\mu_0(s_t))f(\mu_0(s_t))-1+F(\mu_0(s_t))\leq 0$$

$$\implies \frac{\partial}{\partial s_t}\mu_1(s_t)\leq 0$$

This establishes that $\mu_1(s_t)$ is decreasing.

By the first order condition for μ_1, we get the following

$$\mu_1(s_t)\ (\ 1+(1-s_t)\ \lambda(s_t)\ )-\frac{1-F(\mu_1(s_t))}{f(\mu_1(s_t))}=1+(\ \mu_1(s_t)-s_t\ )\ \lambda(s_t) \tag{4}$$

By the binding obedience constraint, we get that $\mu_0(1)=\mu_1(1)$. Using this and evaluating equation (4) for $s_t=1$ we get

$$\mu_1(1)-\frac{1-F(\mu_1(1))}{f(\mu_1(1))}=0 \tag{5}$$

By assumption **??**, $\frac{1-F(\mu)}{f(\mu)}$ is convex and we can apply Jensen inequality to claim the following

$$\mathbf{E}[\ \mu\ ]=\mathbf{E}\left[\frac{1-F(\mu)}{f(\mu)}\right]\geq\frac{1-F(\mathbf{E}[\ \mu\ ])}{f(\mathbf{E}[\ \mu\ ])}$$

$$\implies \mathbf{E}[\ \mu\ ]-\frac{1-F(\mathbf{E}[\ \mu\ ])}{f(\mathbf{E}[\ \mu\ ])}\geq 0$$

Moreover, $\mu-\frac{1-F(\mu)}{f(\mu)}$ is increasing by assumption 1. Equation (5) then implies that $\mu_0(1)=\mu_1(1)\leq\mathbf{E}[\ \mu\ ]$. Monotonicity of $\mu_0(s_t)$ then establishes $\mu_0(s_t)<\mathbf{E}[\ \mu\ ]$ for all s_t.

The designer responds to a marginal increase in standards by increasing μ_0 and lowering μ_1. As $\mu_0<\frac{-v_l}{v_h-v_l}$, the average ability of applicants excluded by the designer is low enough. In particular, equation (3) holds for $s_t'=1$ and for all $s_t\in\left[\frac{-v_l}{v_h-v_l},1\right)$.

To complete the argument, we need to show that $\mu_0(s_t) > 0$ for all $s_t > \mathbf{E}[\mu]$. This rules out the corner solution with $\mu_0(s_t) = 0$. As $s_t > \mathbf{E}[\mu]$, by obedience constraint we get

$$\mu_0(s_t) = 0 \implies \mu_1(s_t) < 1$$

Using the first-order necessary conditions, we get

$$\frac{1}{s_t} \geq \frac{1}{s_t}\left[1 + \frac{1 - F(\mu_1(s_t))}{(1 - \mu_1(s_t))f(\mu_1(s_t))}\right]$$

This leads to a contradiction as $1 + \frac{1-F(\mu_1(s_t))}{(1-\mu_1(s_t))f(\mu_1(s_t))} > 1$. This also establishes the required lower bound in section A.5 ($\mu_0^* > 0$).

□

A.7 Equilibrium Well-posedness

We first show that in any pure strategy equilibrium the top employer's standard $s_t \geq s_b$. Assume for contradiction that $s_t < s_b$, then the designer chooses to allocate to only the top employer. If $s_t < s_b$, any test allocation ρ is revenue equivalent to a perturbed test allocation $\hat{\rho}$. Where $\hat{\rho}$ is defined such that for all μ and θ, the allocation $\hat{\rho}$ recommends the applicant to the top employer with probability $\rho_\theta^t(\mu) + u\rho_\theta^b(\mu)$ and rejects with probability $1 - \rho_\theta^t(\mu) - u\rho_\theta^b(\mu)$. This introduces slack to the top employers' obedience constraint, as $s_b > s_t$, thus ρ is not optimal. By section A.6, the top employer has a profitable deviation of setting $s_t = s_b$. This shows there is no equilibrium (in pure strategy) for which $s_t < s_b$.

Now we will prove the continuity of the employer payoffs, which will allow us to establish the existence of a mixed strategy equilibrium.

Lemma 2: Let F satisfy assumption 1. For each $(s_t, s_b) \in \left[\frac{-v_l}{v_h - v_l}, 1\right]^2$, the employer payoffs U_t and U_b are constant over the set of solutions to the primal problem (P). Moreover, the payoffs U_t and U_b vary continuously with (s_t, s_b).

Proof. I present the proof in two parts. First, I will establish the conditions for Berge's Maximum Theorem (see chapter 17 in Aliprantis and Border (2006) for reference). This will establish the upper hemicontinuity continuity of the set of optimal solutions to (P). Second, I will show that for any given $s = (s_t, s_b)$, the employers' utility U_t and U_b are constant over the set of optimal mechanisms. This, along with upper hemicontinuity of the solution set and contunity of U_t, U_b in the test allocation ρ implies continuity of U_t and U_b in s.

Claim 1: Conditions for Berge's Maximum Theorem (Theorem 17.31 in Aliprantis and Border (2006)) hold.

Recall the set of feasible tests is $\mathcal{F}$ (see A.4) and that $\mathcal{F}$ is bounded, convex, and weakly compact. The objective function is continuous on the feasible set. Thus, to use Berger's Maximum theorem, we need to establish the continuity of the following set-valued map representing the obedience constraints.

Define the obedience constraint correspondence by

$$\mathcal{O}(s_t, s_b) := \left\{ \rho \in \mathcal{F} \;\middle|\; \begin{array}{l} (1-s_t)\int_0^1 \mu\rho_h^t(\mu)\, dF(\mu) - s_t \int (1-\mu)\rho_l^t(\mu)\, dF(\mu) = 0, \\ (1-s_b)\int_0^1 \mu\rho_h^b(\mu)\, dF(\mu) - s_b \int (1-\mu)\rho_l^b(\mu)\, dF(\mu) = 0 \end{array} \right\}$$

and its graph

$$\mathrm{Gr}(\mathcal{O}) := \left\{ (s,\rho) \mid s \in \left[\frac{-v_l}{v_h - v_l}, 1\right]^2,\; \rho \in \mathcal{O}(s) \right\}$$

The correspondence $\mathcal{O}$ has a closed graph because the equalities defining the set $\mathcal{O}(s)$ are jointly continuous in s and ρ. As $\mathcal{O}$ is compact-valued, the above implies that $\mathcal{O}$ is upper hemicontinous. Lower hemicontinuity follows from considering lotteries over test allocations

$$\hat{\rho} = (1 - \varepsilon_t - \varepsilon_b - \delta_t - \delta_b)\cdot\rho \;+\; \varepsilon_t\cdot(1,0,0,0) \;+\; \varepsilon_b\cdot(0,1,0,0) \;+\; \delta_t\cdot(1,0,1,0) \;+\; \delta_b\cdot(0,1,0,1)$$

For a given $s = (s_t, s_b) \in \left(\frac{-v_l}{v_h - v_l}, 1\right)^2$. Note that for any open set $Y \subset \mathcal{F}$ such that $\rho \in Y$ there exists arbitrarily small $\varepsilon_t, \varepsilon_b, \delta_t, \delta_b > 0$ for which $\hat{\rho} \in Y$. In particular, there exists an open set $\mathrm{Nb}(s)$ such that $s \in \mathrm{Nb}(s)$ and $Y \cap \mathcal{O}(s') \neq \varnothing$ for all $s' \in \mathrm{Nb}(s)$. For boundary values $s_j \in \left\{\frac{-v_l}{v_h - v_l}, 1\right\}$ we set ε_j or δ_j equal to zero, appropriately.

Claim 2: For any given $s = (s_t, s_b)$, the employers' utility U_t and U_b are constant over the set of optimal mechanisms.

Fix some (s_t, s_b). Represent the set of solutions to (R) by $X^*(s)$. Recall for an optimal test allocation ρ the obedience constraint binds. Thus for any $\rho \in X^*(s)$ we get

$$U_t(\rho) = \left(v_h + v_l\frac{1 - s_t}{s_t}\right)\mathbf{E}\left[\,\mu\rho_h^t(\mu)\,\right]$$

$$U_b(\rho) = \left(v_h + v_l\frac{1 - s_b}{s_b}\right)\mathbf{E}\left[\,\mu\rho_h^b(\mu)\,\right]$$

In particular, U_t and U_b are constant over $X^*(s)$ if the following holds

$$\rho, \rho' \in X^*(s) \implies \rho_h = \rho'_h \tag{6}$$

I will show that the above statement (6) holds by means of contradiction. Assume that there exist $\rho, \rho' \in X^*(s)$ such that $\rho_h \neq \rho'_h$. In particular, there exists a set $S \subset [0,1]$ of types with positive measure such that $\rho_h(\mu) \neq \rho'_h(\mu)$ for all $\mu \in S$. Note that $X^*(s)$ is convex so $\alpha\rho + (1-\alpha)\rho' \in X^*(s)$

for all $\alpha \in [0,1]$. We will proceed in two cases:
Case I: There exists a positive probability subset $S' \subset S$ such that for all $\mu \in S'$ either $K(\mu;\rho) > 0$ or $K(\mu;\rho') > 0$. In this case, we consider some $\alpha \in (0,1)$ and define $\hat{\rho} := \alpha\rho + (1-\alpha)\rho'$. On the set S', we have $0 < \hat{\rho}_h^t,\ \hat{\rho}_h^b < 1$ and $K(\,\cdot\,;\,\hat{\rho}) > 0$. The previous statement follows from the characterization of extreme points. We can then choose two positive probability (ordered) subsets $S_1' < S_2' \subset S'$. Finally, we perturb $\hat{\rho}$ on S_1' by slightly increasing $\hat{\rho}_h^b$ and slightly decreasing $\hat{\rho}_h^t$. To maintain obedience, we also perturb $\hat{\rho}$ on S_2' by slightly increasing $\hat{\rho}_h^t$ and slightly decreasing $\hat{\rho}_h^b$. A calculation similar to proof of Proposition 3, in section A.4, shows that constructing such a perturbation without violating obedience is feasible and yields strictly greater revenue, a contradiction.
Case II: For all $\mu \in S$ the slope $K(\mu;\rho) = K(\mu;\rho') = 0$. Define $\hat{\rho} := \alpha\rho + (1-\alpha)\rho'$ and $\overline{\mu} := \inf\{\,\mu \mid K(\mu,\hat{\rho}) > 0\}$. We can partition $[0,\overline{\mu}]$ into intervals $[\mu_i,\mu_{i+1})$ for $0 = \mu_0 \le \mu_1 \le \mu_2 \le \mu_3 \le \mu_4 \le \overline{\mu}$. Where

$$\int_{\mu_4}^{\overline{\mu}} \mu \, dF(\mu) = \alpha \int_0^{\overline{\mu}} \mu \cdot 1_{\{\rho=(1,0,1,0)\}} \, dF(\mu) + (1-\alpha) \int_0^{\overline{\mu}} \mu \cdot 1_{\{\rho'=(1,0,1,0)\}} \, dF(\mu)$$

$$\int_{\mu_3}^{\mu_4} \mu \, dF(\mu) = \alpha \int_0^{\overline{\mu}} \mu \cdot 1_{\{\rho=(0,1,u,0)\}} \, dF(\mu) + (1-\alpha) \int_0^{\overline{\mu}} \mu \cdot 1_{\{\rho'=(0,1,u,0)\}} \, dF(\mu)$$

$$\int_{\mu_2}^{\mu_3} \mu \, dF(\mu) = \alpha \int_0^{\overline{\mu}} \mu \cdot 1_{\{\rho=(u,0,0,1)\}} \, dF(\mu) + (1-\alpha) \int_0^{\overline{\mu}} \mu \cdot 1_{\{\rho'=(u,0,0,1)\}} \, dF(\mu)$$

$$\int_{\mu_1}^{\mu_2} \mu \, dF(\mu) = \alpha \int_0^{\overline{\mu}} \mu \cdot 1_{\{\rho=(0,1,0,1)\}} \, dF(\mu) + (1-\alpha) \int_0^{\overline{\mu}} \mu \cdot 1_{\{\rho'=(0,1,0,1)\}} \, dF(\mu)$$

The partition constructed above rearranges the mass of high ability ($\theta = h$) applicants for types below $\overline{\mu}$. Define $\widetilde{\rho}$ which equals $\hat{\rho}$ for types greater than $\overline{\mu}$ and defined as following for types $\mu \le \overline{\mu}$

$$\widetilde{\rho}(\mu) = \begin{cases} (0,0,0,0) & \mu < \mu_1 \\ (0,1,0,1) & \mu \in [\mu_1,\mu_2) \\ (u,0,0,1) & \mu \in [\mu_2,\mu') \\ (u,0,u,0) & \mu \in [\mu',\mu_3) \\ (0,1,u,0) & \mu \in [\mu_3,\mu_4) \\ (1,0,1,0) & \mu \in [\mu_4,\overline{\mu}] \end{cases} \quad \text{or} \quad \begin{cases} (0,0,0,0) & \mu < \mu_1 \\ (0,1,0,1) & \mu \in [\mu_1,\mu') \\ (0,1,u,0) & \mu \in [\mu',\mu_2) \\ (u,0,u,0) & \mu \in [\mu_2,\mu_3) \\ (0,1,u,0) & \mu \in [\mu_3,\mu_4) \\ (1,0,1,0) & \mu \in [\mu_4,\overline{\mu}] \end{cases}$$

Where μ' is chosen such that

$$\int_0^{\overline{\mu}} (1-\mu) \cdot \left(1_{\{\widetilde{\rho}=(0,1,0,1)\}} + 1_{\{\widetilde{\rho}=(u,0,0,1)\}}\right) \, dF(\mu)$$
$$= \alpha \int_0^{\overline{\mu}} (1-\mu) \cdot 1_{\{\rho=(0,1,0,1)\}} \, dF(\mu) + (1-\alpha) \int_0^{\overline{\mu}} (1-\mu) \cdot 1_{\{\rho'=(0,1,0,1)\}} \, dF(\mu)$$

By construction, we get

$$\int_0^1 (1-\mu)\widetilde{\rho}_l^b(\mu) \, dF(\mu) = \int_0^1 (1-\mu)\hat{\rho}_l^b(\mu) \, dF(\mu)$$

and

$$\int_0^1 \mu\widetilde{\rho}_h^j(\mu)\, dF(\mu) = \int_0^1 \mu\hat{\rho}_h^j(\mu)\, dF(\mu)$$

for $j \in \{t, b\}$.

When $\rho \neq \rho'$, the allocation $\widetilde{\rho}$ induces slack in the top employer's obedience constraint. The test allocation $\widetilde{\rho}$ is constructed such that the mass of low ability applicants hired by the bottom employer is preserved, and the mass of low ability applicants hired by the top employer is pushed forward to higher types. As $(1-\mu)$ is decreasing, the test allocation $\widetilde{\rho}$ leads to a strictly lower mass of low ability applicants hired by the top employer relative to $\hat{\rho}$. We can construct a perturbation $\rho'' := \beta\widetilde{\rho} + (1-\beta)(1,0,1,0)$ for $\beta \in (0,1)$. We can choose β so that the top employer's obedience constraint binds, leading to a contradiction as the allocation ρ'' yields a greater revenue than $\hat{\rho}$. □

By the Glicksberg's theorem, see Fudenberg and Tirole (1991), continuity of the first period payoffs and compact choice sets implies the existence of an equilibrium with possibly mixed strategies played by the employers.

A.8 Proof of Theorem 2

Theorem 2: Let F be uniformly distributed on $[0,1]$. There exists constants $C_2 > C_1 > 1/2$ such that if $C_1 \leq u < \frac{-v_l}{v_h - v_l} < C_2$, then σ is part of an equilibrium only if $\sigma_t(1) = 0$, i.e. in equilibrium the top employer sets standard $s_t = 1$ with zero probability.

Proof. Fix some mixed strategy profile $\sigma = (\sigma_t, \sigma_b)$. First note that it is without loss to focus on equilibrium where $\inf \operatorname{supp}(\sigma_b)$ and $\inf \operatorname{supp}(\sigma_t)$ are greater than the reservation expected ability $\frac{-v_l}{v_h - v_l}$.

For uniform distribution on $[0,1]$ we get that $\frac{1-F(\mu)}{(1-\mu)f(\mu)} = 1$. The certifier's objective (see equation (2) and section (A.6)) can be represented by

$$\mathbf{E}\left[\left(\frac{2\mu}{s_t} - 1\right)\rho_h^t(\mu) + u\left(\frac{2\mu}{s_b} - 1\right)\rho_h^b(\mu)\right] \tag{7}$$

If $\sup \operatorname{supp}(\sigma_b) \leq \frac{2u}{1+u}$, then for all $s_b \in \operatorname{supp}(\sigma_b)$ and $s_t = 1$ the line $u\left(\frac{2\mu}{s_b} - 1\right)$ lies strictly above the line $2\mu - 1$ for $\mu \in [0,1]$. From equation (7), we observe that if $\sup \operatorname{supp}(\sigma_b) \leq \frac{2u}{1+u}$ and $s_t = 1$ then the certifier only allocates the applicant to the bottom employer. In particular, for $s_t = 1$ the top employer gets a payoff of zero against the bottom employer's strategy σ_b. This implies the top employer plays $s_t = 1$ with zero probability as otherwise the top employer has a profitable deviation by playing $s_t' = \inf \operatorname{supp}(\sigma_b) + \varepsilon$ for some small $\varepsilon > 0$.

Now consider σ_b such that $\sup\ (\sigma_b) > \frac{2u}{1+u}$. In perticular $\sigma_b\ \left(\ \left(\frac{2u}{1+u}, 1\right]\ \right) > 0$. As $\sup \operatorname{supp}(\sigma_b) > \frac{2u}{1+u}$ the strategy σ_b puts positive probability on actions s_b for which the lines $u\left(\frac{2\mu}{s_b} - 1\right)$ and $2\mu - 1$

intersect. The point of intersection is given by $\mu = \frac{(1-u)s_b}{2(s_b-u)}$. More generally if the lines $u\left(\frac{2\mu}{s_b}-1\right)$ and $\frac{2\mu}{s_t}-1$ intersect on the interval $[0,1]$ then the intersection point is given by $\frac{(1-u)s_t s_b}{2(s_b-us_t)}$. Note that the intersection point $\frac{(1-u)s_t s_b}{2(s_b-us_t)}$ is strictly increasing in s_t.

The bottom employer never plays $s_b = 1$ with positive probability in equilibrium, as $s_b = 1$ yields a payoff of 0 to the bottom employer for any standards chosen by the top employer. If $\sigma_b(1) > 0$ then the bottom employer's total payoff is zero as the bottom employer must be indifferent between all strategies in support of σ_b. Thus, a mixed strategy σ_b with $\sigma_b(1) > 0$ is part of an equilibrium only if $\sigma_t\left(\frac{-v_l}{v_h-v_l}\right) = 1$ as otherwise the bottom employer has a profitable deviation of playing $s_b' = \frac{-v_l}{v_h-v_l} + \delta$ for some $\delta > 0$. But the top employer never plays $s_t = \frac{-v_l}{v_h-v_l}$ with probability 1 in equilibrium. Thus $\sigma_b(1) = 0$.

Consider any $s_b \in \left(\frac{2u}{1+u}, 1\right)$ and $s_t \in [1-\varepsilon, 1]$ for small $\varepsilon > 0$, point wise maximizing equation (7) results in the following allocation

$$(\,\rho_h^t(\mu),\, \rho_h^b(\mu)\,) = \begin{cases} (0,0) & \mu \in [0, s_b/2) \\ (0,1) & \mu \in \left[s_b/2, \frac{(1-u)s_t s_b}{2(s_b-us_t)}\right) \\ (1,0) & \mu \in \left[\frac{(1-u)s_t s_b}{2(s_b-us_t)}, 1\right] \end{cases}$$

The total mass of low ability applicant that can be pooled in under this point wise optimal allocation is

$$\int_{s_b/2}^{1} (1-\mu)\, d\mu$$

Using binding obedience constraint we get that the total mass of low ability applicants that is needed for obedience is given by the following

$$\frac{1-s_b}{s_b}\int_{s_b/2}^{\frac{(1-u)s_t s_b}{2(s_b-us_t)}} \mu\, d\mu + \frac{1-s_t}{s_t}\int_{\frac{(1-u)s_t s_b}{2(s_b-us_t)}}^{1} \mu\, d\mu$$

The first term is for low ability applicants required for bottom employer's obedience constraint and the second term is for the top employer's obedience constraint. For $s_t = 1$, the pointwise optimal test allocation is obedient whenever the following inequality holds

$$\int_{s_b/2}^{1} (1-\mu)\, d\mu \;-\; \frac{1-s_b}{s_b}\int_{s_b/2}^{\frac{(1-u)s_b}{2(s_b-u)}} \mu\, d\mu \;> 0$$

Simplifying the above expression, we get

$$(2-s_b)^2 - \frac{(1-u)^2 s_b(1-s_b)}{(s_b-u)^2} + s_b(1-s_b) \geq 0$$

Equivalently

$$(4-3s_b)(s_b-u)^2 - (1-u)^2 s_b(1-s_b) \geq 0$$

The above is strictly greater than 0 when $s_b \geq \frac{2u}{1+u}$ and $u > 1/2$. Let $C_1 = 0.51$ and $u \geq 0.51$, under this choice of u the pointwise optimal test allocation is obedient for all $s_b \in \left(\frac{2u}{1+u}, 1\right)$ and $s_t = 1$. Moreover, because of the strict inequality for small enough $\varepsilon > 0$ the pointwise optimal test allocation is obedient for all $s_b \in \left(\frac{2u}{1+u}, 1\right)$ and $s_t \in [1-\varepsilon, 1]$.

To complete the proof, we will show that there is some value of $\frac{-v_l}{v_h - v_l} > C_1$ such that deviating from $s_t = 1$ to $s_t = 1 - \varepsilon$ is profitable for the top employer against any bottom employer standard $s_b \in \left(\frac{2u}{1+u}, 1-\varepsilon\right)$, where $\varepsilon > 0$ is small enough. Combining this with the fact that $\sigma_b\left(\left(\frac{2u}{1+u}, 1\right)\right) > 0$ implies that in equilibrium $\sigma_t(1) = 0$.

Consider $s_b \in \left(\frac{2u}{1+u}, 1-\varepsilon\right)$, where $\varepsilon > 0$ is small enough such that the point wise optimal test is obedient and $\sigma_b(\,[1-\varepsilon, 1]\,) = 0$. The requirement $\sigma_b(\,[1-\varepsilon, 1]\,) = 0$ is an equilibrium restriction on σ_b; we will verify this formally after constructing the top employer's profitable deviation. If the top employer deviates from $s_t = 1$ to $s_t = 1 - \varepsilon$ then the intersection point of the coefficients in front of ρ_h^t and ρ_h^b in equation (7) shift from $\frac{(1-u)s_b}{2(s_b-u)}$ to $\frac{(1-u)(1-\varepsilon)s_b}{2(s_b-u(1-\varepsilon))}$. As $\frac{(1-u)s_b}{2(s_b-u)} > \frac{(1-u)(1-\varepsilon)s_b}{2(s_b-u(1-\varepsilon))}$, the optimal mechanism allocates more applicant types to the top employer under $s_t = 1 - \varepsilon$ when compared to $s_t = 1$.

Let H_0, L_0 represent the mass of high and low ability applicants allocated to the top employer by the pointwise optimal mechanism for $s_t = 1$, respectively. By obedience $L_0 = 0$ and as argued previously

$$H_0 = \int_{\frac{(1-u)s_b}{2(s_b-u)}}^{1} \mu \, d\mu$$

Let H_1, L_1 be the additional mass of high and low ability applicants added to the top employer, respectively, when $s_t = 1 - \varepsilon$. By binding obedience, we get

$$\frac{H_0 + H_1}{H_0 + H_1 + L_1} = 1 - \varepsilon$$

$$\implies L_1 = \frac{\varepsilon}{1-\varepsilon}(H_0 + H_1)$$

Combining the above, we get that the expected ability of the additional applicants added to the top employer is given by

$$\frac{H_1}{H_1 + L_1} = \frac{(1-\varepsilon)H_1}{H_1 + \varepsilon H_0}$$

By pointwise maximization of equation (7) we get the following

$$H_1 = \int_{\frac{(1-u)(1-\varepsilon)s_b}{2(s_b-u(1-\varepsilon))}}^{\frac{(1-u)s_b}{2(s_b-u)}} \mu \, d\mu$$

$$= \frac{(1-u)^2 s_b^2}{8}\left[\frac{1}{(s_b-u)^2} - \frac{(1-\varepsilon)^2}{(s_b - u(1-\varepsilon))^2}\right]$$

For $s_t = 1-\varepsilon$ to be a profitable deviation the expected ability of the added applicant must exceed the reservation expected ability. To prove the theorem, it suffices to show that there is a constant C_2 such that $C_1 < C_2 \le \frac{(1-\varepsilon)H_1}{H_1+\varepsilon H_0}$.

We can evaluate $\frac{(1-\varepsilon)H_1}{H_1+\varepsilon H_0}$ at $\varepsilon = 0$ using L'Hôpital's rule. This yields

$$\left.\frac{-H_1 + (1-\varepsilon)\frac{d}{d\varepsilon}H_1}{H_0 + \frac{d}{d\varepsilon}H_1}\right|_{\varepsilon=0}$$

Plugging in $\frac{d}{d\varepsilon}H_1 = \frac{(1-u)^2 s_b^3(1-\varepsilon)}{4(s_b-u(1-\varepsilon))^3}$ and values of H_0 and H_1 we get the following

$$\left.-H_1 + (1-\varepsilon)\frac{d}{d\varepsilon}H_1\right|_{\varepsilon=0} = \frac{(1-u)^2 s_b^3}{4(s_b-u)^3}$$

and

$$\left.H_0 + \frac{d}{d\varepsilon}H_1\right|_{\varepsilon=0} = \frac{1}{2} + \frac{(1-u)^2 s_b^2(s_b+u)}{8(s_b-u)^3}$$

Putting together, we get

$$\left.\frac{-H_1 + (1-\varepsilon)\frac{d}{d\varepsilon}H_1}{H_0 + \frac{d}{d\varepsilon}H_1}\right|_{\varepsilon=0} = \frac{2(1-u)^2 s_b^3}{4(s_b-u)^3 + (1-u)^2 s_b^2(s_b+u)}$$

Inverting the expression above, we get the following

$$\frac{1}{2} + \frac{u}{2s_b} + \frac{2(s_b-u)^3}{(1-u)^2 s_b^3}$$

Taking the derivative with respect to s_b yields

$$u\frac{12(s_b-u)^2 - (1-u)^2 s_b^2}{2(1-u)^2 s_b^4}$$

The expression above is positive for $s_b \ge \frac{2u}{1+u}$, thus $\frac{1}{2} + \frac{u}{2s_b} + \frac{2(s_b-u)^3}{(1-u)^2 s_b^3}$ is increasing in s_b. Plugging in $s_b = 1$ we get that the above expression is upper bounded by 1.8 as $u \ge C_1 > 1/2$.

This implies that $\left.\frac{-H_1+(1-\varepsilon)\frac{d}{d\varepsilon}H_1}{H_0+\frac{d}{d\varepsilon}H_1}\right|_{\varepsilon=0} > 0.55$. By continuity of $\frac{(1-\varepsilon)H_1}{H_1+\varepsilon H_0}$ in ε and the intermediate value theorem we get that for small enough $\varepsilon > 0$ the expected ability of additional applicants $\frac{(1-\varepsilon)H_1}{H_1+\varepsilon H_0} \ge 0.53 > 1/2$. We establish the theorem by setting $C_2 = 0.53$ and $C_1 = 0.51$.

To finish the proof, we need to show that the equilibrium restriction on the support of σ_b is valid. For this, we will show that in equilibrium $s_b = 1 \notin \text{supp}(\sigma_b)$. In other wrds we show that for small enough $\varepsilon > 0$ the equilibrium strategy σ_b assigns zero probability to the set $[1-\varepsilon, 1]$, i.e. $\sigma_b\left(\,[1-\varepsilon, 1]\,\right) = 0$.

Consider some $s_b \in (1-\varepsilon, 1]$, by the point wise maximization of equation (7), we observe that the bottom employer gets zero payoff against any $s_t \le s_b$. Thus, we only need to show that the bottom employer has a profitable deviation by playing $1-\varepsilon$ instead of s_b against any $s_t > s_b$. Given some hiring standards $s_b \in (1-\varepsilon, 1)$ and $s_t > s_b$ for small $\varepsilon > 0$, the point wise maximization of equation (7) and binding obedience constraints yield the following payoff to the bottom employer

$$\begin{aligned} U_b(\rho) &= \left(v_h + v_l \frac{1-s_b}{s_b}\right) \mathbf{E}\left[\ \mu \rho_h^b(\mu)\ \right] \\ &= (v_h - v_l)\left(1 + \frac{v_l}{s_b(v_h - v_l)}\right) \int_{s_b/2}^{\frac{(1-u)s_t s_b}{2(s_b - u s_t)}} \mu \, d\mu \end{aligned}$$

When $\frac{-v_l}{v_h - v_l} < s_b$ the term $\left(1 + \frac{v_l}{s_b(v_h - v_l)}\right)$ is positive. By differentiating $\int_{s_b/2}^{\frac{(1-u)s_t s_b}{2(s_b - u s_t)}} \mu \, d\mu$ with respect to s_b we see that the expression is decreasing in s_b. Thus, when the bottom employer lowers its standard from s_b to $1-\varepsilon$, the certifier allocates additional mass of applicants to the bottom employer. The deviation is profitable if and only if the expected ability of new applicants allocated to the bottom employer exceeds $\frac{-v_l}{v_h - v_l}$.

Let N_0, M_0 represent the mass of high and low ability applicants that are allocated to the bottom employer given standards (s_t, s_b), respectively. From point wise maximization of equation (7) and binding obedience constraint, we get the following

$$N_0 = \int_{s_b/2}^{\frac{(1-u)s_t s_b}{2(s_b - u s_t)}} \mu \, d\mu$$

and

$$M_0 = \frac{1-s_b}{s_b} N_0$$

Similarly, for standards $(s_t, 1-\varepsilon)$ let N_1, M_1 be the additional mass of high and low ability applicants that are allocated to the bottom employer, respectively.

$$N_1 = \int_{(1-\varepsilon)/2}^{s_b/2} \mu \, d\mu + \int_{\frac{(1-u)s_b s_t}{2(s_b - u s_t)}}^{\frac{(1-u)(1-\varepsilon)s_t}{2((1-\varepsilon) - u s_t)}} \mu \, d\mu$$

and

$$\begin{aligned} \frac{N_0 + N_1}{N_0 + M_0 + N_1 + M_1} &= 1 - \varepsilon \\ \implies M_1 &= \left(\frac{\varepsilon}{1-\varepsilon} - \frac{1-s_b}{s_b}\right) N_0 + \frac{\varepsilon}{1-\varepsilon} N_1 \end{aligned}$$

Thus, the expected ability of the additional mass of applicants added to the bottom employer is given by

$$\frac{N_1}{N_1 + M_1} = \frac{N_1(1-\varepsilon)s_b}{(s_b + \varepsilon - 1)N_0 + s_b N_1}$$

Fix any $s_b \in (1-\varepsilon, 1)$. To show that the bottom employer has a profitable deviation by playing $1-\varepsilon$, it suffices to show that the right-hand side of the above expression is greater than $\frac{-v_l}{v_h - v_l}$ for all $s_t > s_b$ and sufficiently small $\varepsilon > 0$.

The expected ability of the additional applicants can be bounded below by some constant $0 < C' < 1$ and some small $\varepsilon > 0$ if the following holds

$$\frac{N_1(1-\varepsilon)s_b}{(s_b+\varepsilon-1)N_0 + s_b N_1} \geq C'$$

After rearranging, we get

$$(1-\varepsilon-C')N_1 \geq C'\left(1-\frac{1-\varepsilon}{s_b}\right)N_0$$

Differentiating N_1 with respect to s_t shows N_1 is increasing in s_t as $\frac{s_b^3}{(s_b-us_t)^3}$ is decreasing in s_b. Differentiating N_0 with respect s_t shows that N_0 is increasing in s_t as $\frac{(1-u)s_t s_b}{2(s_b-us_t)}$ is increasing in s_t. By plugging in $s_t = s_b$ for N_1 and $s_t = 1$ for N_0 we get the following implication

$$(1-\varepsilon-C')\int_{(1-\varepsilon)/2}^{\frac{(1-u)(1-\varepsilon)s_b}{2(1-\varepsilon-us_b)}} \mu\, d\mu \geq C'\left(1-\frac{1-\varepsilon}{s_b}\right)\int_{s_b/2}^{\frac{(1-u)s_b}{2(s_b-u)}} \mu\, d\mu$$
$$\implies (1-\varepsilon-C')N_1 \geq C'\left(1-\frac{1-\varepsilon}{s_b}\right)N_0$$

As $\int_{s_b/2}^{\frac{(1-u)s_b}{2(s_b-u)}} \mu\, d\mu$ is decreasing in s_b the first inequality above holds for any constant $0.6 > C' > 0$ if the following stronger condition holds

$$(0.4-\varepsilon)\int_{(1-\varepsilon)/2}^{\frac{(1-u)(1-\varepsilon)s_b}{2(1-\varepsilon-us_b)}} \mu\, d\mu \Bigg/ \left[\left(1-\frac{1-\varepsilon}{s_b}\right)\int_{(1-\varepsilon)/2}^{\frac{(1-u)(1-\varepsilon)}{2(1-\varepsilon-u)}} \mu\, d\mu\right] \geq C'$$

Notice that both the numerator and the denominator of the expression on the left-hand side tend to zero as $s_b \to 1-\varepsilon$. Taking the derivative with respect to s_b of the numerator and the denominator of the fraction on the left-hand side of the inequality above.

$$\frac{\partial}{\partial s_b}(0.4-\varepsilon)\int_{(1-\varepsilon)/2}^{\frac{(1-u)(1-\varepsilon)s_b}{2(1-\varepsilon-us_b)}} \mu\, d\mu = (0.4-\varepsilon)\frac{(1-u)^2(1-\varepsilon)^3 s_b}{4(1-\varepsilon-us_b)^3}$$

and

$$\frac{\partial}{\partial s_b}\left(1-\frac{1-\varepsilon}{s_b}\right)\int_{(1-\varepsilon)/2}^{\frac{(1-u)(1-\varepsilon)}{2(1-\varepsilon-u)}} \mu\, d\mu = \frac{(1-\varepsilon)^3}{8s_b^2}\left(\left(\frac{1-u}{1-\varepsilon-u}\right)^2 - 1\right)$$

The derivative of the numerator is increasing in s_b and the derivative of the denominator is decreasing in s_b. This implies the following inequalities

$$\frac{\partial}{\partial s_b}(0.4-\varepsilon)\int_{(1-\varepsilon)/2}^{\frac{(1-u)(1-\varepsilon)s_b}{2(1-\varepsilon-us_b)}} \mu\, d\mu \geq (0.4-\varepsilon)\frac{1-\varepsilon}{4(1-u)}$$

and

$$\frac{\partial}{\partial s_b}\left(1-\frac{1-\varepsilon}{s_b}\right)\int_{(1-\varepsilon)/2}^{\frac{(1-u)(1-\varepsilon)}{2(1-\varepsilon-u)}} \mu\, d\mu \leq \frac{(1-\varepsilon)}{8}\left(\left(\frac{1-u}{1-\varepsilon-u}\right)^2 - 1\right)$$

By L'Hopital's rule and the two inequalities above, we get the following

$$\lim_{s_b \to 1-\varepsilon} \frac{(0.4-\varepsilon)\int_{(1-\varepsilon)/2}^{\frac{(1-u)(1-\varepsilon)s_b}{2(1-\varepsilon-us_b)}} \mu \, d\mu}{\left(1-\frac{1-\varepsilon}{s_b}\right)\int_{(1-\varepsilon)/2}^{\frac{(1-u)(1-\varepsilon)}{2(1-\varepsilon-u)}} \mu \, d\mu} \geq \frac{2(0.4-\varepsilon)(1-\varepsilon-u)^2}{(1-u)(2-\varepsilon-2u)\varepsilon} > 0$$

The right-hand side blows up for ε close to zero. Thus, the necessary inequality, $(1-\varepsilon-C')N_1 \geq C'\left(1-\frac{1-\varepsilon}{s_b}\right)N_0$, holds for some $0.6 > C' > C_2 = 0.53$ when ε is small enough. This establishes the required restriction on the support of σ_b, in equilibrium, and concludes the proof. □

Remark. The argument in proof of Theorem 2 can be extended to show a slightly stronger claim that in equilibrium $s_t = 1 \notin \text{supp}(\sigma_t)$. I do not pursue this in the interest of brevity.